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Contribution to Stability Control of Nonlinear Systems

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Abstract. The most powerful methods of systems analysis have been developed for linear control systems. For a linear control system, all the relationships between the variables are linear differential equations, usually with constant coefficients. Actual control systems usually contain some nonlinear elements. In the following we show how the equations for nonlinear systems may be linearized. But the result is only applicable in a sufficiently small region in the neighbourhood of equilibrium point.

The table in this paper includes the nonlinear equations and their the linear approximation. Then it is easy to find out if the nonlinear system is or is not stable; the task that usually ranks among the difficult tasks in engineering practice.

Keywords: nonlinear system, equilibrium points, phase-plane trajectory

1. Introduction

A nonlinear autonomous n-order system is considered. This system may be described by one nonlinear n-order equation or by a set on n first-order nonlinear differential equations

$$\begin{aligned} x_1' &= f_1(x_1, x_2, \dots, x_n) \\ x_2' &= f_2(x_1, x_2, \dots, x_n) \\ &\dots\dots\dots\dots \\ x_n' &= f_n(x_1, x_2, \dots, x_n) \end{aligned} \tag{1}$$

or matrix equation

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \tag{2}$$

The solution of the system (1) is given phase-plane trajectory in the n-dimensional state space. The points of the space in which is $f_1(\mathbf{x}) = f_2(\mathbf{x}) = \dots f_n(\mathbf{x}) = 0$ are singular points of the system because in the equilibrium points are speeds $x_1' = x_2' = \dots x_n' = 0$.

The matrix representation for the linear system (1), where $\mathbf{f}(\mathbf{x})$ is linear function $\mathbf{x}$ we can write $\mathbf{x}' = \mathbf{A}\mathbf{x}$ supposing that $\det \mathbf{A} \neq 0$ is solution $\mathbf{x} = 0$. The linear time-invariant system has an equilibrium point at the origin.

A nonlinear system can have more equilibrium points because $\mathbf{f}(\mathbf{x}) = 0$ can have more solutions – more singular points. The equilibrium points can be stable or unstable; this depends on the phase-plane trajectory. They are stable if the trajectory approaches the equilibrium point as t tends to infinity and they are unstable if the trajectory recedes.

A stability theory plays a central role in the systems theory and engineering. Stability of an equilibrium points can be found out by linearization of the equations (1) in the neighbourhood of each equilibrium point and then it is necessary to find out stability of a surrogate system. If the linearization is allowable then the nonlinear system behaves similarly as the linearized system in the neighbourhood of equilibrium point.

If we can express function f_i in a set (1) in Taylor series in the neighbourhood of each singular point, then we can write for this singular point

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1 - x_{10}) = \left(\frac{\partial f_1}{\partial x_1}\right)(x_1 - x_{10}) + \ldots + \left(\frac{\partial f_1}{\partial x_n}\right)(x_n - x_{n0})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{3}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(x_n - x_{n0}) = \left(\frac{\partial f_n}{\partial x_1}\right)(x_1 - x_{10}) + \ldots + \left(\frac{\partial f_n}{\partial x_n}\right)(x_n - x_{n0})$$

or matrix way

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x} - \mathbf{x}_0) = \mathbf{J}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \tag{4}$$

where

$$\mathbf{J}(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{5}$$

is Jacobian matrix that is defined as the matrix of partial derivatives with numerical values given singular point. The equation (3) is a set of linear differential equations that substitute the original set (1).

A necessary and sufficient condition of stability of the system is that the characteristic equation has all the roots in the left half-plane. If the characteristic equation has one or more roots in the right half-plane, the system is unstable. If the single or multiple roots are located on the imaginary axis, we cannot find out stability using linearization. However this stability that was found out by linearization is only applicable in a sufficiently small region in the neighbourhood of equilibrium point.

2. Linearization of second or third order system

Now we will accomplish the practical linearization of the second or third order system assuming the equations

$$y'' + g(y') + f(y) = 0 \qquad y''' + h(y'') + g(y') + f(y) = 0 \qquad (6) - (7)$$

If this is rearranged as two first-order or third-order equations, choosing the phase variables as the state variables, that is $x_1 = y$; $x_2 = y'$ or $x_1 = y$; $x_2 = y'$; $x_3 = y'''$, then equations (6) or (7) can be written as

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -g(x_2) - f(x_1) \end{aligned} \qquad \begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -h(x_3) - g(x_2) - f(x_1) \end{aligned} \qquad (8) - (9)$$

where is $f_1 = x_2$; $f_2 = -g(x_2) - f(x_1)$ for second order and and $f_1 = x_2$; $f_2 = x_3$; $f_3 = -h(x_3) - g(x_2) - f(x_1)$ for third order and the functions are a special case of equation (1).

Singular points of this system will be obtained by solving $f_1 = f_2 = 0$ or $f_1 = f_2 = f_3 = 0$ and they are the points on the real axis $[x_{10}\,;0]$ or $[x_{10}\,;0\,;0]$. When we introduce for conciseness the symbol

$$\psi = -g(x_2) - f(x_1) \qquad \xi = -h(x_3) - g(x_2) - f(x_1) \qquad (10) - (11)$$

then the Jacobian matrix of this system is (where we have to give for $x_1\,; x_2$ the coordinates of singular point $x_1 = x_{10}\,; x_2 = 0$ or for $x_1\,; x_2\,; x_3$ the coordinates $x_1 = x_{10}\,; x_2 = 0\,; x_3 = 0$)

$$\mathbf{J}(\mathbf{x}_0) = \begin{bmatrix} 0 & 1 \\ \frac{\partial \psi}{\partial x_1} & \frac{\partial \psi}{\partial x_2} \end{bmatrix} \qquad \mathbf{J}(\mathbf{x}_0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\partial \xi}{\partial x_1} & \frac{\partial \xi}{\partial x_2} & \frac{\partial \xi}{\partial x_3} \end{bmatrix} \qquad (12) - (13)$$

and then the following equations represent the linearization of the nonlinear equations about the equilibrium point second order:

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1 - x_{10}) = x_2 \qquad x_1' = x_2$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_2 = \frac{\partial\psi}{\partial x_1}(x_1 - x_{10}) + \frac{\partial\psi}{\partial x_2}x_2 \qquad x_2' = \frac{\partial\psi}{\partial x_1}x_1 - \frac{\partial\psi}{\partial x_1}x_{10} + \frac{\partial\psi}{\partial x_2}x_2 \tag{14}$$

third order:

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1 - x_{10}) = x_2 \qquad x_1' = x_2$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_2 = x_3 \qquad x_2' = x_3 \tag{15}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_3 = \frac{\partial\xi}{\partial x_1}(x_1 - x_{10}) + \frac{\partial\xi}{\partial x_2}x_2 + \frac{\partial\xi}{\partial x_3}x_3 \qquad x_3' = \frac{\partial\xi}{\partial x_1}x_1 - \frac{\partial\xi}{\partial x_1}x_{10} + \frac{\partial\xi}{\partial x_2}x_2 + \frac{\partial\xi}{\partial x_3}x_3$$

These equations correspond to the linearized second order equation

$$y'' - \frac{\partial\psi}{\partial x_2}y' - \frac{\partial\psi}{\partial x_1}y + \frac{\partial\psi}{\partial x_1}x_{10} = 0 \tag{16}$$

and to the linearized third order equation

$$y''' - \frac{\partial\xi}{\partial x_3}y'' - \frac{\partial\xi}{\partial x_2}y' - \frac{\partial\xi}{\partial x_1}y + \frac{\partial\xi}{\partial x_1}x_{10} = 0 \tag{17}$$

We can substitute the original nonlinear equations (6) – (7) by these equations (16) – (17) and to find out stability of the nonlinear system like stability of the linear system, but only in a sufficiently small region in the neighbourhood of equilibrium point.

Let us notice that the absolute members in the equations (16) – (17) are constants that do not influence stability. As long as a singular point lies at the origin $x_{10} = x_{20} = 0$ or $x_{10} = x_{20} = x_{30} = 0$, the constants are zero and the equation (16) is

$$y'' - \frac{\partial\psi}{\partial x_2}y' - \frac{\partial\psi}{\partial x_1}y = 0 \tag{18}$$

and the equation (17) is

$$y''' - \frac{\partial\xi}{\partial x_3}y'' - \frac{\partial\xi}{\partial x_2}y' - \frac{\partial\xi}{\partial x_1}y = 0 \tag{19}$$

For example to find out stability of the second order system

$$y'' + \left(1 + 3y^2\right)y' + (2 + 5y)y = 0$$

Substitute into this $x_1 = y$; $x_2 = y'$ the equation will be

$$x_1' = x_2$$
$$x_2' = -x_2 - 3x_1^2x_2 - 2x_1 - 5x_1^2$$

The system has too equilibrium points. First point is at the origin $[0\,;0]$. In this point is

$$\frac{\partial\psi}{\partial x_1} = \frac{\partial\left(-x_2 - 3x_1^2x_2 - 2x_1 - 5x_1^2\right)}{\partial x_1} = -6x_1x_2 - 2 - 10x_1\Big|_{[0;0]} = -2$$
$$\frac{\partial\psi}{\partial x_2} = \frac{\partial\left(-x_2 - 3x_1^2x_2 - 2x_1 - 5x_1^2\right)}{\partial x_2} = -1 - 3x_1^2\Big|_{[0;0]} = -1$$

and the linearized equation (18) is

$$y'' + y' + 2y = 0$$

The roots are negative - the equilibrium point is stable. The system is stable in the neighbourhood of this equilibrium point $[0\,;0]$.

The second equilibrium point is $[-0{,}4\,;0]$. In this point is

$$\frac{\partial \psi}{\partial x_1} = 2\,; \qquad \frac{\partial \psi}{\partial x_2} = -1{,}48\,;$$

and the linearized equation (18) is

$$y'' + 1{,}48 y' - 2y - 0{,}8 = 0$$

The roots of characteristic equation are positive - the equilibrium point is unstable. The system is unstable in the neighbourhood of this equilibrium point $[-0{,}4\,;0]$.

For example to find out stability of the third order system

$$y''' + 2(1 + y')y'' + \left(1 - y^2\right)y' + y + 3y^3 = 0$$

Substitute into this $x_1 = y\,;\ x_2 = y'\,; x_3 = y''$ the equation will be

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = -2x_3 - 2x_2x_3 - x_2 + x_1^2 x_2 - x_1 - 3x_1^3$$

The system has a unique equilibrium point at the origin $[0\,;0\,;0]$. In this point is

Tab.1: Linearized equations of nonlinear systems

	equation of nonlinear system	singular point	linearized equation
1	$y'' + \left(d + ey + fy^2\right)y' + (p + qy)y = 0$	$[0;0]$	$y'' + dy' + py = 0$
		$\left[-\frac{p}{q};0\right]$	$y'' - \left[d - \frac{p}{q}\left(e - f\frac{p}{q}\right)\right]y' - py - \frac{p^2}{q} = 0$
2	$y'' + \left(d + ey + fy^2\right)y' + \left(p + ry^2\right)y = 0$	$[0;0]$	$y''' + ay'' + dy' + py = 0$
3	$y''' + (a + by)y'' + (d + ey)y' + (p + qy)y = 0$	$[0\,;0\,;0]$	$y''' + ay'' + dy' + py = 0$
		$\left[-\frac{p}{q};0;0\right]$	$y''' - \left(a - b\frac{p}{q}\right)y'' - \left(d - e\frac{p}{q}\right)y' - py - \frac{p^2}{q} = 0$
4	$y''' + \left(a + by + cy^2\right)y'' + \left(d + ey + fy^2\right)y' + (p + qy)y = 0$	$[0\,;0\,;0]$	$y''' + ay'' + dy' + py = 0$
		$\left[-\frac{p}{q};0;0\right]$	$y''' - \left[a - \frac{p}{q}\left(b - c\frac{p}{q}\right)\right]y'' - \left[d - \frac{p}{q}\left(e - f\frac{p}{q}\right)\right]y' - py - \frac{p^2}{q} = 0$
5	$y'' + \left(d + ey + fy^2\right)y' + \left(g + hy + ky^2\right)y'^2 + (p + qy)y = 0$	$[0;0]$	$y'' + dy' + py = 0$
		$\left[-\frac{p}{q};0\right]$	$y'' - \left[d - \frac{p}{q}\left(e - f\frac{p}{q}\right)\right]y' - py - \frac{p^2}{q} = 0$
6	$y''' + \left(a + by + cy^2\right)y'' + \left(d + ey + fy^2\right)y' + \left(p + ry^2\right)y = 0$	$[0\,;0\,;0]$	$y''' + ay'' + dy' + py = 0$
7	$y''' + \left(a + by + cy^2\right)y'' + \left(d + ey + fy^2\right)y' + \left(g + hy + ky^2\right)y'^2 + (p + qy)y = 0$	$[0\,;0\,;0]$	$y''' + ay'' + dy' + py = 0$
		$\left[-\frac{p}{q};0;0\right]$	$y''' - \left[a - \frac{p}{q}\left(b - c\frac{p}{q}\right)\right]y'' - \left[d - \frac{p}{q}\left(e - f\frac{p}{q}\right)\right]y' - py - \frac{p^2}{q} = 0$

$$\frac{\partial \xi}{\partial x_1} = 2x_1x_2 - 1 - 9x_1^2\Big|_{[0;0;0]} = -1$$

$$\frac{\partial \xi}{\partial x_2} = -2x_3 - 1 + x_1^2\Big|_{[0;0;0]} = -1$$

$$\frac{\partial \xi}{\partial x_3} = -2 - 2x_2\Big|_{[0;0;0]} = -2$$

and the linearized equation (18) is

$$y''' + 2y'' + y' + y = 0 \qquad \mathrm{H}_2 = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 > 0$$

The characteristic equation has positive coefficients and the Hurwitz determinant is positive too. The equilibrium point is stable. The system is stable in the neighbourhood of this equilibrium point.

Table 1 shows the commonly used equations of the second and third order, their singular points and the linearized equation for the neighbourhood of equilibrium point. The table helps us to find out immediately if the equilibrium point is stable or unstable.

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