ISBN 978-1-84626-xxx-x
Proceedings of 2011 International Conference on Optimization of the Robots and Manipulators (OPTIROB 2011)
Sinaia, Romania, 26-28 Mai, 2011, pp. xxx-xxx

Study of Circular Tapping Plates, with a Circumferential Vertical Load at the Interior Boundary by Transfer-Matrix Method

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Abstract. The work presents a calculus of the circular tapping plates, embedded at the exterior circumference, charged with a concentrated circumferential vertical load at the interior circumference, using the Transfer-Matrix Method. The approach is based of the theory of Dirac's and Heaviside's functions and operators. The circular plate's calculus is important for a lot of industry domains, including the robotics too. We can obtain the two state vectors: for the first ring-the exterior circumferential element and for the latest ring-the interior circumferential element. We can calculate after, for the values $r_0<r<R$, all the state vectors for all the rings of the circular tapping plate. With the Transfer-Matrix Method is very easy to program this calculus algorithm.

Keywords: Dirac's function, Heaviside's function, state vector, Transfer-Matrix, circular tapping plate, circumferential vertical load.

1. Introduction

The elasticity theory of the circular plates is very hard and has developed in [2], based of some hypothesis.

For this work, we have a Transfer-Matrix, for the circular plate, charged with an exterior load density q(x,y), written by the Dirac's and Heaviside's functions and operators [1]. The load density function should be integrated in a general manner for the differential equation. After integration, we obtain the deformed of the circular plates. Now, we can express, for the application, the boundary conditions for the displacements calculus or for the stresses calculus in any plate points.

2. Assumptions Calculus

The bending plate calculus theory is based on some simplifying hypothesis [1]: we admit that the points aligned on the same normal line at the median surface before deformation remains aligned on the same normal line at the deformed surface after deformation; the normal stresses in the parallel sections at the median plan are negligible compared with the bending stresses; it not exist crush for the plate layers some more than others, besides that local crash under a concentrated load; the sign convention is as follows: the positive arrow is to up and the positive angle is counterclockwise; when the arrow w decreases, the angle ω is negative and $\frac{dw}{dr}$ is negative too.

3. Fundamental Differential Equations of the Circular Plates

Wee have a circular plate loaded with an exterior density load q(x,y). The deformation of any plate is given by the general equation:

$$\Delta\Delta w(x,y) = \frac{1}{D} q(x,y) \tag{1}$$

$\Delta\Delta$ is the double Laplacian operator, with the expression:

$$\Delta\Delta = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \tag{2}$$

$$D = \frac{Eh^3}{12\left(1 - \nu^2\right)} \tag{3}$$

and E is the longitudinal elasticity modulus, ν is the Poisson's coefficient and h is the plate thickness.

We can integrate the equation (1), function essentially on the contour integration domain. We have solution only for the circular plates with axially symmetric loads. We will establish the equations which allow to calculate the efforts and the deformations in all points of the circular plate. We consider the arrow w(r) of the circular plate at the distance r from the axis and ω is the angle around which it spins the normal line (Fig. 1., a.).

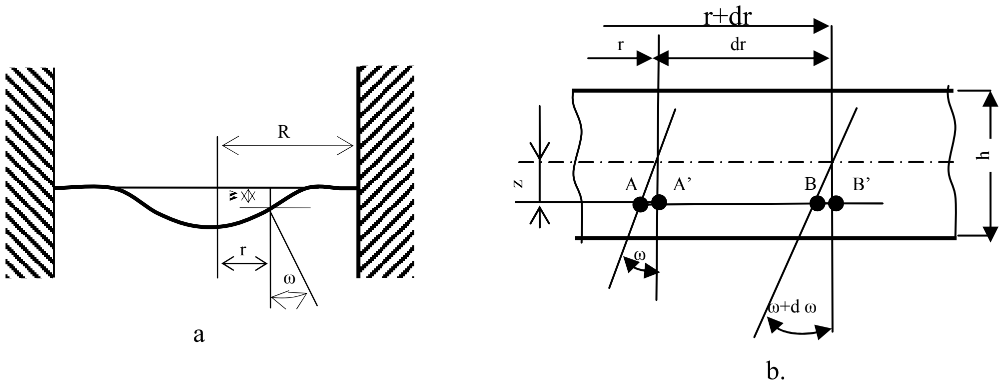

Figure 1. a. Circular deformed plate; b. Axial section of the plate.

The arrow w(r) and the angle ω(r) is given by:

$$\omega = \frac{dw}{dr} \tag{4}$$

The Figure 1., b. shows one axial section and two normal sections of the plate, at distance between them of r-before deformation and r+dr-after deformation. At the rate z of the middle of the fiber, the relative elongation of segment AB is:

$$A'B' - AB = z(\omega + d\omega) - z\omega = zd\omega \tag{5}$$

and the relative radial elongation is:

$$\varepsilon_r = z\frac{d\omega}{dr} \tag{6}$$

A' is on the circle of radius r+zω. The tangential relative elongation is:

$$\varepsilon_t = \frac{2\pi \cdot z\omega}{2\pi \cdot r} = z\frac{\omega}{r} \tag{7}$$

The deformations and the radial and tangential stresses are linked by the following formulas [2] of the elasticity theory:

$$\begin{cases} \varepsilon_r = \dfrac{1}{E}(\sigma_r - \nu\sigma_t) \\ \varepsilon_t = \dfrac{1}{E}(\sigma_t - \nu\sigma_r) \end{cases} \tag{8}$$

and vice versa [2]:

$$\begin{cases} \sigma_r = \dfrac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_t) \\ \sigma_t = \dfrac{E}{1-\nu^2}(\varepsilon_t + \nu\varepsilon_r) \end{cases} \tag{9}$$

The expression (9), with (6) and (7), is:

$$\begin{cases} \sigma_r = \dfrac{Ez}{1-\nu^2}\left(\dfrac{d\omega}{dr} + \nu\dfrac{\omega}{r}\right) \\ \sigma_t = \dfrac{Ez}{1-\nu^2}\left(\dfrac{\omega}{r} + \nu\dfrac{d\omega}{dr}\right) \end{cases} \tag{10}$$

We are cut a sector prism element on the plate [1]. The balance equations are:

$$\begin{cases} M_r r d\varphi = r d\varphi \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_r z dz = \dfrac{Eh^3}{12(1-\nu^2)}\left(\dfrac{d\omega}{dr} + \nu \dfrac{\omega}{r}\right) r d\varphi \\ M_t dr = dr \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_t z dz = \dfrac{Eh^3}{12(1-\nu^2)}\left(\dfrac{\omega}{r} + \nu \dfrac{d\omega}{dr}\right) dr \end{cases} \tag{11}$$

when M_r is the radial moment and M_t is the tangential moment, the moments are applied on the element faces and they are taken per unit length after one radial axis. If the stresses σ_r and σ_t is known, we can to calculate the resultant of these moments on the faces. We have with (3):

$$\begin{cases} M_r = D\left(\dfrac{d\omega}{dr} + \nu \dfrac{\omega}{r}\right) \\ M_t = D\left(\dfrac{\omega}{r} + \nu \dfrac{d\omega}{dr}\right) \end{cases} \tag{12}$$

with notation:

$$\mathrm{M}_{res} = \frac{1}{D(1+\nu)}(M_r + M_t) \tag{13}$$

After calculus, we have:

$$\mathrm{M}_{res} = \frac{\omega}{r} + \frac{d\omega}{dr} = \frac{1}{r}\frac{d\omega}{dr} + \frac{d^2\omega}{dr^2} = \frac{1}{r}\frac{d}{dr}(r\omega) \tag{14}$$

M_{res} is proportional at a bending moment and the cutting force T is per unit of length, on a circumference of radius r. After the vertical axis, the balance equation for the radial prism element (Figure 2., b.) is:

$$(T + dT)(r + dr)d\varphi - Trd\varphi + q(r)\cdot rdr \cdot d\varphi = 0 \tag{15}$$

where:

$$\frac{d}{dr}(rT) = -q(r)\cdot r \tag{16}$$

The balance equation, for the element, in report with the tangential axis of the arc circle of radius r, at the level of median plane, writing the sum of the moments is:

$$(M_r + dM_r)(r + dr)d\varphi - M_r r d\varphi + q(r)\cdot rdr \cdot d\varphi \frac{dr}{2} - M_t dr \cdot d\varphi + (T + dT)(r + dr)dr \cdot d\varphi = 0 \tag{17}$$

The small higher order term is neglected. After we have:

$$M_t - \frac{d}{dr}(rM_r) = rT \tag{18}$$

In (18), we replace M_r and M_t with (13) and we obtain:

$$\frac{1}{D}rT = \frac{\omega}{r} - r\frac{d^2\omega}{dr^2} - \frac{d\omega}{dr} = -r\frac{d\mathrm{M}_{res}}{dr} \tag{19}$$

We note:

$$\tau = \frac{1}{D}rT \tag{20}$$

The deformations and the efforts in a circular plate loaded with an axially symmetrical force are, after the differential base equations are:

$$\begin{cases} \dfrac{d\tau}{dr} = -\dfrac{1}{D}rq(r) \\ \dfrac{d\mathrm{M}_{res}}{dr} = -\dfrac{\tau}{r} \\ \dfrac{d(r\omega)}{dr} = r\mathrm{M}_{res} \\ \dfrac{dw}{dr} = \omega \end{cases} \tag{21}$$

In (21) we have four sizes [1]: the displacements-w is the arrow, ω is the angle, with a physic significance; M_{res} is proportional at a bending moment and τ is related at the cut force by the relation (20).

For (21), after integration, we have four constants: w_R, ω_R, M_{resR} and τ_R and:

$$\begin{cases} \tau = \tau_R - q_1(r) \\ \mathrm{M}_{res} = \mathrm{M}_{resR} - \tau_R \log\dfrac{r}{R} + q_2(r) \end{cases} \tag{22}$$

with the notations:

$$\begin{cases} q_1(r) = \int_R^r \dfrac{\rho}{D} q(\rho) d\rho \\ q_2(r) = \int_R^r \dfrac{1}{\rho} q_1(\rho) d\rho \end{cases} \tag{23}$$

With (21) and (22), we can write:

$$\frac{d(r\omega)}{dr} = \mathrm{M}_{resR}\, r - \tau_R r \log\frac{r}{R} + r q_2(r) \tag{24}$$

and after integration, we obtain:

$$\omega = \omega_R \frac{R}{r} + \frac{1}{2}\mathrm{M}_{resR}\left(r - \frac{R^2}{r}\right) - \frac{1}{2}\tau_R\left(r \log\frac{r}{R} + \frac{R^2}{2r} - \frac{r}{2}\right) + \frac{1}{r} q_3(r) \tag{25}$$

when:

$$q_3(r) = \int_R^r \rho q_2(\rho) d\rho \tag{26}$$

The arrow w is:

$$w = w_R + \omega_R R \log\frac{r}{R} + \frac{1}{2}\mathrm{M}_{resR}\left(\frac{r^2 - R^2}{2} - R^2 \log\frac{r}{R}\right) - \frac{1}{2}\tau_R\left(\frac{r^2 + R^2}{2}\log\frac{r}{R} + \frac{r^2 - R^2}{2}\right) + q_4(r) \tag{27}$$

with:

$$q_4(r) = \int_R^r \frac{1}{\rho} q_3(\rho) d\rho \tag{28}$$

4. Circular Plate with its Transfer-Matrix

We define a state vector with four elements, for a circular plate, at the radius r: $\{V(r)\}_r = \{\mathrm{M}_{res}(r), \tau(r), w(r), \omega(r)\}^{-1}$ and we have the state vector for the radius r_0: $\{V(r_0)\}_0 = \{\mathrm{M}_{res0}, \tau_0, w_0, \omega_0\}^{-1}$ and we have the state vector for the radius R: $\{V(R)\}_R = \{\mathrm{M}_{resR}, \tau_R, w_R, \omega_R\}^{-1}$ .For the radius r we can write: $\{V(r)\}_r = [T]_r \{V(R)\}_R + \{V_e\}_r$ with: [T]r as the transfer-matrix for the circular plate at the radius r and $\{V_e\}_r$ is the vector for the exterior loads.

5. Application: Circular Tapping Plate, Embedded at the Exterior Circumference with a Circumferential Vertical Load at the Interior Boundary

In the Figure 3 we have a tapping circular plate, embedded at the exterior boundary and free at the interior boundary, charged with circumferential vertical load at the interior circumference.

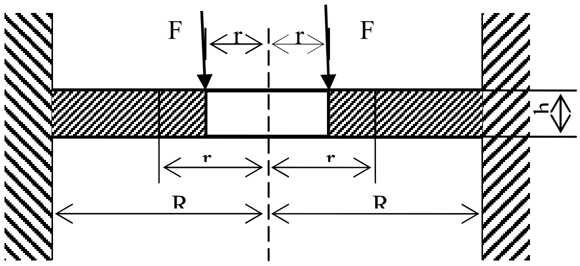

Figure 3. A circular tapping plate, embedded at the exterior boundary, with circumferential vertical load.

With Dirac's and Heaviside's functions and operators, the associated function for the circumferential vertical load, the charge density, is [1]:

$$q(r) = -F\delta(r - r_0) \tag{29}$$

The functions $q_i(r)$, $i=1,4$, for the concentrated circumferential vertical load at the interior boundary are:

$$q_1(r)=\frac{Fr_0}{D}[1-Y(r-r_0)] \tag{30}$$

$$q_2(r)=-\frac{Fr_0}{D}[1-Y(r-r_0)]\log\frac{r_0}{r} \tag{31}$$

$$q_3(r)=\frac{Fr_0}{2D}[1-Y(r-r_0)]\left(\frac{r_0^2-r^2}{2}-\frac{r^2}{2}\log\frac{r_0}{r}\right) \tag{32}$$

$$q_4(r)=\frac{Fr_0}{4D}[1-Y(r-r_0)]\left[r_0^2-r^2-\left(r_0^2+r^2\right)\log\frac{r_0}{r}\right] \tag{33}$$

For $r=r_0$, the matrix relation (32) becomes:

$$\{V(r_0)\}_0=[T]_{r_0}\{V(R)\}_R+\{V_e\}_{r_0} \tag{34}$$

when:

$$[T]_{r_0}=\begin{bmatrix} 1 & -\log\frac{r_0}{R} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2}\left(\frac{r_0^2-R^2}{2}-R^2\log\frac{r_0}{R}\right) & -\frac{1}{4}\left[\left(r_0^2+R^2\right)\log\frac{r_0}{R}+\left(R^2-r_0^2\right)\right] & 1 & R\log\frac{r_0}{R} \\ \frac{1}{2}\left(r_0-\frac{R^2}{r_0}\right) & -\frac{1}{2}\left(r_0\log\frac{r_0}{R}+\frac{R^2}{2r_0}-\frac{r_0}{2}\right) & 1 & \frac{R}{r_0} \end{bmatrix} \tag{35}$$

and: $q_1(r)=\frac{r_0}{D}F$; $q_2(r)=0$, $q_3(r)=0$, $q_4(r)=0$.

The boundary conditions are: for the exterior circumference embedded (r=R): $w_R=0$, $\omega_R=0$ and for the interior free circumference ($r=r_0$), with the circumferential vertical load F: $\tau_0=\frac{r_0}{D}F$, $M_{res}(r_0)=0$.

For M_{res0}, the expression (13) becomes:

$$M_{res0}=\frac{1}{1+\nu}\left[\frac{\omega_0}{r_0}+\nu\left(\frac{d\omega}{dr}\right)_{r_0}\right] \tag{36}$$

For M_t, we have:

$$M_t=D\frac{\omega_0}{r_0}\left(1-\nu^2\right) \tag{37}$$

After calculus, we obtain:

$$M_{res0}=\frac{1-\nu}{r_0}\omega_0 \tag{38}$$

Now, we have the matrix equation:

$$\begin{Bmatrix} \frac{1-\nu}{r_0}\omega_0 \\ \frac{r_0}{D}F \\ w_0 \\ \omega_0 \end{Bmatrix}=[T]_{r_0}\begin{Bmatrix} M_{resR} \\ \tau_R \\ 0 \\ 0 \end{Bmatrix}+\begin{Bmatrix} q_2(r_0) \\ -q_1(r_0) \\ q_4(r_0) \\ \frac{1}{r_0}q_3(r_0) \end{Bmatrix} \tag{39}$$

This is a linear equations system, with four unknowns in four linear equations. This system is very easy to solve and we obtain the four solutions for w_0, ω_0, M_{resR} and τ_R. After, we can calculate in all sections of the tapping plate, for $r<r_0<R$, the four elements for each state vector for all sections.

6. Conclusions

This paper presents an application for the theoretical calculus with the Transfer-Matrix Method for a tapping circular plate, embedded at the exterior boundary, free at the interior circumference, loaded with a circumferential vertical charge at the interior boundary. The matrix calculus is very easy to program and so, that is very important for optimization calculus, function of different optimization criteria.

We intend, in the future, to extend this calculus for the thin plates, continuous or tapping and for the membrane and diaphgram calculus, the is important for the study and the calculus of the diaphragm pumps.

7. References

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