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Some Solutions of the Modified Korteweg-de Vries Equation by Painleve Test

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Abstract. The Korteweg-de Vries (KdV) equation which is a third order diffusion equation has been of interest since John Scott Russell (1845) [1]. In this work we study the modified Korteweg-de Vries (mKdV) equation, through this study we find that the (mKdV) does not satisfy Painleve's property even although that we were able to find analytic solution for this equation.

Keywords: Kortewege-de Vrise equation, Painleve's property, Modified Korteweg-de Vries equation, Resonanse points , Exact solutions.

1. Introduction

Most phenomena in the many scientific fields can be classified as linear or nonlinear differential equation, these phenomena can be described as for instance, plasma or flow of blood, or the movement of water waves as well [2]. In this work, we tried to find a solution to this kind of equations although it is normally not easy to find clear-cut solutions. But by using Painleve's method it is possible to help finding a clear solution which physicists, doctors, and others may benefit from.

2. Painleve Test

In the present section we apply Painleve's equation in the (mKdV) equation:

$$u_t + \alpha u^2 u_x + u_{xxx} = 0, \qquad \alpha \in \mathbb{R} \setminus \{0\} \tag{1}$$

Let $u = \sum_{j=0}^{\infty} u_j \varphi^{j-p}$ be the perfect series solution of the equation (1), where u_j and φ are analytic functions in a neighborhood of the manifold $\varphi = \varphi(t,x) = 0$ [3].

First, we need to calculate value of p, where p is the equipoise point in the series solution. Now, to derive u in the series solution, where $u_t(t,x) = \partial u(t,x)/\partial t$, $u_x(t,x) = \partial u(t,x)/\partial x$ and $u_{xxx}(t,x) = \partial^3 u(t,x)/\partial x^3$, by substituting these into the equation (1), and by paralleling the lowest powers in the generated equation we find $j - 3p - 1$ and $j - p - 3$ then $p = 1$, and by accomplishing the summation, we obtain:

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$$\sum_{j=3}^{\infty} u_{j-3,t}\varphi^{j-4} + \sum_{j=2}^{\infty}(j-3)u_{j-2}\varphi_t\varphi^{j-4}$$

$$+\alpha\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}\sum_{i=0}^{k} u_{j-k}u_{k-i}u_i(j-k-1)\varphi_x + \sum_{k=0}^{j-1}\sum_{i=0}^{k} u_{j-k-1,x}u_{k-i}u_i\right]\varphi^{j-4}$$

$$+\sum_{j=3}^{\infty} u_{j-3,xxx}\varphi^{j-4}\sum_{j=2}^{\infty}3(j-3)u_{j-2,xx}\varphi_x\varphi^{j-4} + \sum_{j=2}^{\infty}3(j-3)u_{j-2,x}\varphi_{xx}\varphi^{j-4} \quad (2)$$

$$+\sum_{j=1}^{\infty}3(j-2)(j-3)u_{j-1,x}\varphi_x^2\varphi^{j-4} + \sum_{j=1}^{\infty}3(j-2)(j-3)u_{j-1}\varphi_x\varphi_{xx}\varphi^{j-4}$$

$$+\sum_{j=2}^{\infty}(j-3)u_{j-2}\varphi_{xxx}\varphi^{j-4} + \sum_{j=0}^{\infty}(j-1)(j-2)(j-3)u_j\varphi_x^3\varphi^{j-4} = 0,$$

To acquire u_0, then at $j = 0$ in the equation (2), we obtain:

$$u_0 = i\sqrt{\frac{6}{\alpha}}\varphi_x, \qquad i = \sqrt{-1}, \quad (3)$$

To acquire u_1, then at $j = 1$ in the equation (2), we obtain:

$$u_1 = -\frac{i}{2}\sqrt{\frac{6}{\alpha}}\frac{\varphi_{xx}}{\varphi_x}, \quad (4)$$

Since $p = 1$, by using the technique of amputating, and let $u_j = 0$ for all j > 1.

Then the series solution $u = \frac{1}{\varphi^p}\sum_{j=0}^{\infty} u_j\varphi^j$ becomes:

$$u = \frac{u_0}{\varphi} + u_j \quad (5)$$

To acquire u_2 ,then at $j = 2$ in the equation (2), we obtain:

$$u_2 = -\frac{i}{\sqrt{6\alpha}\varphi_x}\left[\frac{\varphi_t + \varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2\right], \quad (6)$$

Now, in equation (2), we must find all coefficients of u_j
where $u_j = 0$ for all j < 0. $u_j = 0$

$$If \quad k=0 \quad \Rightarrow \alpha\varphi_x\sum_{k=0}^{j}\left[\sum_{i=0}^{k}u_{k-i}u_i\right](j-k-1)u_{j-k} = \alpha\varphi_x u_0^2(j-1)u_j$$

$$If \quad i=k \quad \Rightarrow \alpha\varphi_x\sum_{k=0}^{j}\left[\sum_{i=0}^{k}u_{k-i}u_i\right](j-k-1)u_{j-k} = -\alpha\varphi_x u_0^2 u_j$$

$$If \quad k=j \quad \Rightarrow \alpha\varphi_x\sum_{k=0}^{j}\left[\sum_{i=0}^{k}u_{k-i}u_i\right](j-k-1)u_{j-k} = -\alpha\varphi_x u_0^2 u_j$$

The relation becomes:

$$(j-3)\left[j-\left(\frac{3}{2}\pm\sqrt{\frac{1}{4}-\alpha}\right)\right]\varphi_x^3 u_j = -u_{j-3,t} + \alpha\varphi_x u_0\sum_{i=1}^{j-1}u_{j-i}u_i - (j-3)u_{j-2}\varphi_t$$

$$-\alpha\sum_{k=1}^{j-1}\left[\sum_{i=0}^{k}u_{k-i}u_i\right]u_{j-k}(j-k-1)\varphi_x - \alpha\sum_{k=0}^{j-1}\left[\sum_{i=0}^{k}u_{k-i}u_i\right]u_{j-k-1,x} - u_{j-3,xxx}$$

$$-3(j-3)u_{j-2,x}\varphi_{xx} - (j-3)u_{j-2}\varphi_{xxx} - 3(j-2)(j-3)\left[u_{j-1,x}\varphi_x^2 + u_{j-1}\varphi_x\varphi_{xx}\right]$$

$$-3(j-3)u_{j-2,xx}\varphi_x, \quad (7)$$

We observe that the all coefficients of u_j in the equation (7) are $(j-3)$ and

$\left[j - \left(\frac{3}{2} \pm \sqrt{\frac{1}{4} - \alpha} \right) \right]$, then, in the generic of the integer resonance point is $j = 3$.

The other resonance points are contingent on the value of α. For instance, if $\alpha = -6$, the resonance points are $j = -1, 3, 4$.

Now, at $j = 3$, and by using the equations (3), (4) and (7), we get,

$$-u_{0,t} - u_{0,xxx} + 2\alpha\varphi_x^2 u_1 u_2 - 2\alpha\varphi_x u_0 u_1 u_2 + \alpha u_0^2 u_{2,x} - 2\alpha\varphi_x u_0 u_1 u_{1,x}$$
$$-\alpha u_{0,x} u_1^2 - 2\alpha u_0 u_2 u_{0,x} = 0,$$

but, u_j=0 for all j > 1, we get.

$$u_{0,t} - u_{0,xxx} + 2u_0 u_1 u_{1,x} + \alpha u_{0,x} u_1^2 = 0,$$

Inconsistent at the resonance point $j = 3$, this means that the modified Kortewegde-Vries equation (1), does not satisfy the Painleve's property.

Now, at $j = 4$ in the equation (7), we have,

$$-u_{1,t} - u_{1,xxx} - \varphi_t \varphi_2 - 3\varphi_x u_{2,xx} - 3\varphi_{xx} u_{2,x} - \varphi_{xxx} u_2 - 6\varphi_x^2 u_{3,x} - 6\varphi_x \varphi_{xx} u_3$$
$$+\alpha\varphi_x^3 \sum_{i=1}^{3} u_{4-i} u_i - \alpha\varphi_x \sum_{k=1}^{3} \left[\sum_{i=0}^{k} u_{k-i} u_i \right] (3-k) u_{4-k} + \alpha \sum_{k=0}^{3} \left[\sum_{i=0}^{k} u_{k-i} u_i \right] u_{3-k,x} = \quad (8)$$

By implementing the equation (3) into the equation (8), and $u_j = 0$ for all j > 1,we get,

$$u_{1,t} + \alpha u_1^2 u_{1,x} + u_{1,xxx} = 0, \quad (9)$$

Then u_1 is also a solution of the (mKdV) equation (1).

3. Analytic Solution:

In this section, we pursue the project to derive analytic solution. They are unvarying under this transmutation,

$$T : \varphi \to \frac{a\varphi + b}{c\varphi + d}, \qquad ad \neq bc,$$

The Schwartzian derivative [4].

$$\mathrm{S}(\varphi) = \frac{\varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2, \quad (10)$$

The dimension of velocity,

$$\mathrm{C}(\varphi) = -\frac{\varphi_t}{\varphi_x}, \quad (11)$$

The compatibility of C and S described by [4]:

$$S_t + C_{xxx} + 2C_x S + CS_x = 0. \quad (12)$$

By comparing the equations (10) and (11) with the equation (6), and, u_j=0 for all j>1 we observe:

$$C = S. \quad (13)$$

By substituting $S = C$ into the equation (12), we get:

$$S_t + 3SS_x + S_{xxx} = 0, \quad (14)$$

This is Korteweg-de Vries(KdV) like equation.

4. 4 Exact Solution:

Solution for constant ***S***.

The functions of constant $\mathrm{S} = \pm 2\lambda^2$ where λ is a constant, are solutions of the Korteweg-de Vries like equation (14).

Lemma 1: Let τ_1 and τ_2 be two linearly independent solutions of the equation,

$$\frac{d^2\tau}{dz^2} + f(z) = 0, \qquad (15)$$

which are defined and holomorphic on some simply connected domain *D* in complex plane, then $w(z) = \tau_1(z)/\tau_2(z)$ satisfies the equation [2], [4],

$$\{w, z\} = 2f(z), \qquad (16)$$

Conversely, if *w*(*z*) is a solution of (16) at all points of *D*, then one can find two linearly holomorphic independent solutions τ_1 and τ_2 of (15) such that $w(z) = \tau_1(z)/\tau_2(z)$ in some neighborhood of $z_0 \in D$.

Lemma 2: The Schwartzian derivative is invariant under fractional linear transformation acting on the first argument, the form [2], [4]:

$$\left\{\frac{aw+b}{cw+d}; z\right\} = \{w; z\}, \qquad ad \neq bc,$$

where *a*, *b*, *c* and *d* are constants.

Case I :

For $S=-2\lambda^2$, we get,

$S = \{\varphi, x\} = -2\lambda^2$. Hence $f(x) = -\lambda^2$ in (16), and two linearly independent solutions are:

$$\Psi_1 = E(t)e^{\lambda x} + F(t)e^{-\lambda x} \quad , \quad \Psi_2 = G(t)e^{\lambda x} + H(t)e^{-\lambda x}$$

Therefore by Lemma1 and Lemma 2, obtain:

$$\varphi(t,x) = \frac{E(t)e^{\lambda x} + F(t)e^{-\lambda x}}{G(t)e^{\lambda x} + H(t)e^{-\lambda x}}, \qquad EH \neq FG \qquad (17)$$

By using the equations (11) and (13), then:

$$C = S = -\frac{\varphi_t}{\varphi_x} = -2\lambda^2. \qquad (18)$$

Now, to find the differential equation of coefficients E(t), F(t), G(t) and H(t), we derive $\varphi(t,x)$ in the equation (17), to get $\varphi_t(t,x)$ and $\varphi_x(t,x)$, and substituting them into the equation (18), we obtain:

$$\varphi_t = \frac{\left(G(t)E'(t) - E(t)G'(t)\right)e^{2\lambda x} + \left(H(t)F'(t) - F(t)H'(t)\right)e^{-2\lambda x}}{\left(G(t)e^{\lambda x} + H(t)e^{-\lambda x}\right)^2}$$
$$+ \frac{\left(G(t)F'(t) - F(t)G'(t)\right) + \left(H(t)E'(t) - E(t)H'(t)\right)}{\left(G(t)e^{\lambda x} + H(t)e^{-\lambda x}\right)^2},$$

and

$$\varphi_x = \frac{2\lambda\left(H(t)E(t) - G(t)F(t)\right)}{\left(G(t)e^{\lambda x} + H(t)e^{-\lambda x}\right)^2},$$

Then, the equation (18) becomes:

$$C = \frac{\left(G(t)E'(t) - E(t)G'(t)\right)e^{2\lambda x} + \left(H(t)F'(t) - F(t)H'(t)\right)e^{-2\lambda x}}{-2\lambda\left(H(t)E(t) - G(t)F(t)\right)}$$
$$+ \frac{\left(G(t)F'(t) - F(t)G'(t)\right) + \left(H(t)E'(t) - E(t)H'(t)\right)}{-2\lambda\left(H(t)E(t) - G(t)F(t)\right)} = -2\lambda^2,$$

Then

$$\left(G(t)E'(t)-E(t)G'(t)\right)e^{2\lambda x}+\left(H(t)F'(t)-F(t)H'(t)\right)e^{-2\lambda x}+G(t)F'(t)$$
$$-F(t)G'(t)+\left(H(t)E'(t)-E(t)H'(t)\right)=4\lambda^{3}\left(H(t)E(t)-G(t)F(t)\right)\ ,$$

This takes us to a system of nonlinear ordinary differential equation in all coefficients E(t), F(t),G(t) and H(t), then:

$$(a)\ \ GE'-EG'=0$$
$$(b)\ \ HF'-FH'=0$$
$$(c)\ \ \left(GF'-FG'\right)+\left(HE'-EH'\right)=4\lambda^{3}\left(HE-GF\right)$$

particular solutions of (*a*) and (*b*) respectively are:

$$E(t)=AG(t) \quad \text{and} \quad F(t)=BH(t)$$

where A and B are real arbitrary constants.

By using (*a*), (*b*) and (*c*), we get:

$$B\left(G(t)H'(t)-H(t)G'(t)\right)+A\left(H(t)G'(t)-G(t)H'(t)\right)=4\lambda^{3}H(t)G(t)\left(A-B\right),$$

then;

$$\frac{H'(t)}{H(t)}-\frac{G'(t)}{G(t)}=-4\lambda^{3},$$

By integrating, we get:

$$\frac{H(t)}{G(t)}=\exp\left(-4\lambda^{3}t\right)$$

then the equation (17), becomes:

$$\varphi\left(t,x\right)=\frac{AG(t)\exp\left(\lambda x\right)+BG(t)\exp\left(-4\lambda^{3}t-\lambda x\right)}{G(t)\exp\left(\lambda x\right)+G(t)\exp\left(-4\lambda^{3}t-\lambda x\right)},$$

which leads to:

$$\varphi\left(t,x\right)=\frac{Ae^{\lambda\xi_1}+Be^{-\lambda\xi_1}}{e^{\lambda\xi_1}+e^{-\lambda\xi_1}},\qquad where\quad \xi_1=x+2\lambda^{2}$$
$$=\frac{\left(A+B\right)\cosh\lambda\xi_1+\left(A-B\right)\sinh\lambda\xi_1}{2\cosh\lambda\xi_1},$$

Then:

$$\varphi\left(t,x\right)=K_1+K_2\tanh\lambda\xi_1, \tag{19}$$

where K_1 and K_2 are arbitrary constants, and $K_1=\left(A+B\right)/2$ and $K_2=\left(A-B\right)/2$.

For $K_1=0$, and by substituting the equation (19) into the equation (4), we obtain:

$$u_1=-i\sqrt{\frac{6}{\alpha}}\,\frac{-K_2\lambda^{2}\sec h^{2}\lambda\xi_1\tanh\lambda\xi_1}{K_2\lambda\sec h^{2}\lambda\xi_1},$$

Then:

$$u_1=i\lambda\sqrt{\frac{6}{\alpha}}\tanh\lambda\xi_1,\qquad where\quad \xi_1=x+2\lambda^{2}t.$$

Hence $u_1\left(x,t\right)$ is the first exact solution for modified Korteweg-de Vries equation (1).

Now, by the equations (3), (4), (5) and (19), we obtain:

$$u=\frac{i\sqrt{\dfrac{6}{\alpha}}K_2\lambda\sec h^{2}\lambda\xi_1}{K_2\lambda\tanh\lambda\xi_1}+u_1,$$

Then

$$u = i\lambda\sqrt{\frac{6}{\alpha}}\coth\lambda\xi_1, \qquad where \quad \xi_1 = x + 2\lambda^2 t.$$

Hence *u*(*t*, *x*) is the second exact solution for modified Korteweg-de Vries equation (1).

Case II:

For $S=2\lambda^2$,we have:

$S = \{\varphi, x\} = 2\lambda^2$. Hence $f(x) = \lambda^2$ in (16), and two linearly independent solutions are:

$$\Psi_3 = E(t)e^{\lambda ix} + F(t)e^{-\lambda ix} \quad , \quad \Psi_4 = G(t)e^{\lambda ix} + H(t)e^{-\lambda ix}$$

Therefore, Lemma1 and Lemma 2, obtain:

$$\varphi(t,x) = \frac{E(t)e^{\lambda ix} + F(t)e^{-\lambda ix}}{G(t)e^{\lambda ix} + H(t)e^{-\lambda ix}}, \qquad EH \neq FG, \tag{20}$$

By using the equations (11) and (13), then:

$$C = S = -\frac{\varphi_t}{\varphi_x} = 2\lambda^2. \tag{21}$$

Now, to find the differential equation of coefficients E(t), F(t), G(t) and H(t), we derive $\varphi(t,x)$ in the equation (20), to get $\varphi_t(t,x)$ and $\varphi_x(t,x)$ and by substituting them into the equation(21), we obtain:

$$C = \frac{\left(G(t)E'(t) - E(t)G'(t)\right)e^{2\lambda ix} + \left(H(t)F'(t) - F(t)H'(t)\right)e^{-2\lambda ix}}{-2i\lambda\left(H(t)E(t) - G(t)F(t)\right)}$$

$$+\frac{\left(G(t)F'(t) - F(t)G'(t)\right) + \left(H(t)E'(t) - E(t)H'(t)\right)}{-2i\lambda\left(H(t)E(t) - G(t)F(t)\right)} = 2\lambda^2.$$

This takes us to a system of nonlinear ordinary differential equations in all coefficients *E*(*t*), *F*(*t*),*G*(*t*) and *H*(*t*), then:

$$(a) \quad GE' - EG' = 0$$

$$(b) \quad HF' - FH' = 0$$

$$(c) \quad \left(GF' - FG'\right) + \left(HE' - EH'\right) = -4i\lambda^3\left(HE - GF\right)$$

particular solutions of (*a*) and (*b*) respectively are:

$$E(t) = MG(t) \quad \text{and} \quad F(t) = NH(t)$$

where *M* and *N* are real arbitrary constants.

By substituting these into (c), we get:

$$\frac{H(t)}{G(t)} = \exp\left(4i\lambda^3 t\right)$$

Then the equation (20), becomes:

$$\varphi(t,x) = \frac{MG(t)\exp(\lambda ix) + NG(t)\exp\left(4\lambda^3 it - \lambda ix\right)}{G(t)\exp(\lambda ix) + G(t)\exp\left(4\lambda^3 it - \lambda ix\right)}$$

which leads to:

$$\varphi(t,x) = \frac{Me^{\lambda i\xi_2} + Ne^{-\lambda i\xi_2}}{e^{\lambda i\xi_2} + e^{-\lambda i\xi_2}}, \quad where \quad \xi_2 = x - 2\lambda^2 t,$$

$$= \frac{(M+N)\cos\lambda\xi_2 + (M-N)\sin\lambda\xi_2}{2\cos\lambda\xi_2}.$$

Then:

$$\varphi(t,x) = K_3 + K_4\tan\lambda\xi_2, \tag{22}$$

where K_3 and K_4 are arbitrary constants, and $K_3 = (M+N)/2$ and $K_4 = (M-N)/2$

For $K_3 = 0$,by substituting the equation (22) into the equation (4), we get:

$$\hat{u}_1 = -i\sqrt{\frac{6}{\alpha}}\frac{K_4\lambda^2 \sec^2 \lambda\xi_2 \tan\lambda\xi_2}{K_4\lambda \sec^2 \lambda\xi_2},$$

Then:

$$\hat{u}_1 = -i\lambda\sqrt{\frac{6}{\alpha}}\tan\lambda\xi_2, \quad where \quad \xi_2 = x - 2\lambda^2 t.$$

Hence $\hat{u}_1(x,t)$ is the third exact solution for modified Korteweg-de Vries equation (1).

Now, by the equations (3), (4), (5) and (22), we get:

$$\hat{u} = \frac{i\sqrt{\frac{6}{\alpha}}K_4\lambda \sec^2 \lambda\xi_2}{K_4\lambda \tan\lambda\xi_2} + \hat{u}_1,$$

Then:

$$\hat{u} = i\lambda\sqrt{\frac{6}{\alpha}}\cot\lambda\xi_2 \quad where \quad \xi_2 = x - 2\lambda^2 t.$$

Hence $\hat{u}(x,t)$ is the fourth exact solution for modified Korteweg-de Vries equation (1).

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