

# Application and Analysis of Central Difference Kalman Filter in Celestial Navigation with Disturbances

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**Abstract.** The celestial navigation system is one of the important autonomous navigation systems for spacecraft. And the Unscented Kalman Filter (UKF), which is suitable for nonlinear system, has been widely applied in celestial navigation. However, when the disturbances exist and the state covariance becomes large, UKF will reduce the accuracy and the rapidity of convergence. This paper proposes to employ Central Difference Kalman Filter (CDKF) in celestial navigation with disturbances. The numerical simulation results show that, the navigation performance by CDF is better than UKF, with the error peak value only 30% of that of UKF. The further analysis also shows that CDKF has low sensitivity to state covariance and therefore has better robustness than UKF in the application of celestial navigation with disturbances.

**Keywords:** celestial navigation, central difference kalman filter, disturbance, robustness.

## 1. Introduction

The celestial navigation with the star sensor and horizon sensor fixed on the spacecraft is a sophisticated and reliable autonomous navigation method. With the advantages of providing orbit and attitude information, self-contained and good suitability for space application, celestial navigation has been widely used in many fields [1]. The Extended Kalman Filter (EKF) and Unscented Kalman Filter (UKF) are the two generally employed filter methods to determine the orbit [2], [3].

EKF is the mostly used nonlinear estimating method, it only uses the first order terms in the Taylor series expansion, however, also may introduce large estimation error and instability [4]. EKF acquires the exact information of model and the noise statistics and has the unavoidable error caused by the linearization. Then UKF was proposed by Julier and Uhlmann to solve the above problems [5]. UKF uses the true nonlinear model and a set of sigma sample points produced by the unscented transformation to capture the mean and covariance, and introduce the statistics in Kalman inductive process. UKF requires no linearization and has better accuracy than EKF, however, has the same calculations as EKF. But when the temporary disturbances exist, UKF would induce large estimate error and impact the convergence rapidity due to the wrong state covariance. The Central Difference Kalman Filter (CDKF) further developed uses the Sigma points as UKF, and expands the nonlinear model using Sterling's polynomial interpolation method, avoiding calculating the partial derivative and is suitable for arbitrary functions [6]. CDKF can estimate the states even if the nonlinear function is discontinuous or has singular points, has better accuracy than EKF, and can reach or exceed the accuracy of UKF [7]. Furthermore, CDKF only employs one factor (the half step size  $h$  in central difference) to assure the positive semi-definiteness of the posterior covariance, while UKF uses three factors ( $\alpha$ ,  $\kappa$ ,  $\beta$ ) and constrains them in a range to assure it [8]. Therefore, it has less implementing complexity and better efficiency and filtering stability than UKF, which is the highly recommended filter [6, 9].

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This paper employs CDKF in the orbit determination in celestial navigation for spacecraft with temporary disturbances and wrong state covariance. The numerical simulations are performed using both CDKF and UKF, and then the accuracy and robustness are analyzed and compared.

## 2. Orbit Determination Principle by Direct Horizon

### 2.1. Basic Principles

The basic principle of the orbit determination by directly sensing horizon technique is presented as below, and the observation model is shown in Fig.1.

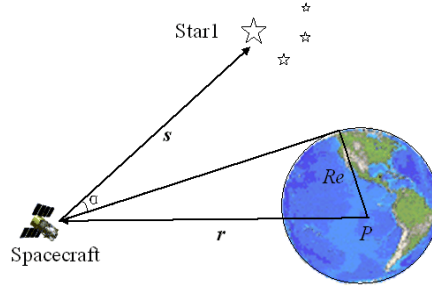


Fig.1: Observation model of celestial navigation method with direct horizon

Identifying the star (e.g. Star1 in Fig.1) using star sensor and measuring the vector of star in star sensor reference frame. Through the transformation matrix of star sensor, the vector of star in the inertial frame ( $\mathcal{S}$ ) is determined. The vertical line of satellite to earth can be measured by earth horizon sensor. Thus the star-elevation angle is obtained as the measurement, is subtended at the spacecraft between the line of sight to the star and earth edge (shown as  $\alpha$  in Fig.1) after the geometrical relations are established, the filter method is employed to real-time estimate the space motion states [10].

### 2.2 Motion Equations

The motion equations used in spacecraft autonomous navigation system, based on orbital dynamics in the earth-centered inertial Cartesian coordinate (J2000.0), is given by the following six equations: [11]

$$\left\{ \begin{array}{l} \frac{dx}{dt} = v_x \\ \frac{dy}{dt} = v_y \\ \frac{dz}{dt} = v_z \\ \frac{dv_x}{dt} = -\mu \frac{x}{r^3} \left[ 1 - J_2 \left( \frac{R_e}{r} \right) \left( 7.5 \frac{z^2}{r^2} - 1.5 \right) \right] + \Delta F_x \\ \frac{dv_y}{dt} = -\mu \frac{y}{r^3} \left[ 1 - J_2 \left( \frac{R_e}{r} \right) \left( 7.5 \frac{z^2}{r^2} - 1.5 \right) \right] + \Delta F_y \\ \frac{dv_z}{dt} = -\mu \frac{z}{r^3} \left[ 1 - J_2 \left( \frac{R_e}{r} \right) \left( 7.5 \frac{z^2}{r^2} - 4.5 \right) \right] + \Delta F_z \end{array} \right. \quad r = \sqrt{x^2 + y^2 + z^2} \quad (1)$$

where the state vector,  $[x, y, z, v_x, v_y, v_z]^T$  are position and velocity of three axis respectively,  $\mu$  is the gravitational constant of the earth,  $r$  is module of the satellite position vector,  $R_e$  is the radius of the earth,  $J_2$  is coefficient of the earth oblateness perturbations,  $[\Delta F_x, \Delta F_y, \Delta F_z]^T$  are the high grade geo-potential, lunar-solar perturbations, radiation pressure and other perturbations of three axis.

Equation (1) is simplified as

$$\dot{\mathbf{X}}(t) = f(\mathbf{X}, t) + \mathbf{w}(t) \quad (2)$$

where  $\mathbf{X} = [x, y, z, v_x, v_y, v_z]^T$ ,  
 $\mathbf{w}(t) = [0, 0, 0, \Delta F_x, \Delta F_y, \Delta F_z]^T$ .

### 2.3 Measurement Equations

As illustrated before, the star-elevation angle is used as measurement, which has been shown in Fig.1 as  $\alpha$ ,

$$\alpha = \arccos\left(-\frac{\mathbf{s} \cdot \mathbf{r}}{r}\right) - \arcsin\left(\frac{R_e}{r}\right) \quad (3)$$

Considering the observation noise, the measurement equations are

$$\begin{aligned} z &= \alpha + v_\gamma \\ &= \arccos\left(-\frac{\mathbf{s} \cdot \mathbf{r}}{r}\right) - \arcsin\left(\frac{R_e}{r}\right) + v_\gamma \end{aligned} \quad (4)$$

where  $\mathbf{r}$  is the satellite position vector in geocentric inertial coordinate system,  $\mathbf{s}$  is unit vector of sight line of the navigation star,  $v_\gamma$  is the measurement noises.

Then the discrete state and measurement equations are

$$\begin{cases} \mathbf{X}_k = F(\mathbf{X}_{k-1}) + \mathbf{w}_k \\ z_k = H(\mathbf{X}_k) + v_k \end{cases} \quad (5)$$

## 3. CDKF Algorithm

CDKF is a sigma-point Kalman filter, employing the second order Sterling's polynomial interpolation formulation for the central divided differences [12]. The algorithm steps contain time update and measurement updates and are described as below.

### 3.1. Time Update Equations

Calculate the Cholesky decomposition of matrix  $P_{k-1|k-1}$ ,

$$P_{k-1|k-1} = S^T S;$$

Set  $\tilde{F}(s) = F(S^T S + x_{k-1|k-1})$  and is approximated by  $\tilde{F}(0) + \sum_{i=1}^n a_i s_i + \frac{1}{2} s^T A s$

where,  $n$  is the number of states,  $h$  is the step size,

$$a_i = \frac{\tilde{F}(he_i) - \tilde{F}(-he_i)}{2h}, 1 \leq i \leq n$$

Complete the time updates

$$x_{k|k-1} = \tilde{F}(0) + \sum_{i=1}^n \frac{1}{2} A_{i,i}$$

$$P_{k|k-1} = Q + \sum_{i=1}^n a_i a_i^T + \sum_{i=1}^n A_{i,i} A_{i,i}^T$$

where  $A_{i,i}$  is the central difference approximation of  $\partial^2 \tilde{F}(s) / \partial s_i^2$ .

### 3.2. Measurement Update Equations

Calculate the Cholesky decomposition of  $P_{k|k-1}$

$$P_{k|k-1} = V^T V$$

Set  $\tilde{H}(s) = H(V^T s + x_{k|k-1})$ , and is approximated by  $\tilde{H}(0) + \sum_{i=1}^n b_i s_i + \frac{1}{2} s^T G s$ ;

Complete the measurement updates

$$x_{k|k} = x_{k|k-1} + L_k (y(k) - z_k)$$

$$P_{k|k} = P_{k|k-1} - L_k P_{xz}^T$$

where

$$z_k = \tilde{H}(0) + \sum_{i=1}^n \frac{1}{2} G_{i,i}$$

$$P_{xz} = V^T (b_1, \dots, b_n)^T$$

$$P_{zz} = \sum_{i=1}^n b_i b_i^T + \sum_{i=1}^n \frac{1}{2} G_{i,i} G_{i,i}^T$$

$$L_k = P_{xz} (R + P_{zz})^{-1}$$

where  $G_{i,i}$  is the central difference approximation of  $\partial^2 \tilde{H}(s) / \partial s_i^2$ .

## 4. Numerical Simulation and Results

### 4.1. Simulation Parameters

#### 4.1.1. Orbit parameters

The initial orbit elements of the spacecraft are: semimajor axis  $a = 7136.635\text{km}$ , eccentricity  $e = 1.809 \times 10^{-5}$ , longitude of the ascending node  $\Omega = 0$ , inclination  $i = 65^\circ$ , argument of periapsis  $\omega = 0$ , radius of the earth  $R_e = 6378.14\text{km}$ . The nominal orbit with the above elements is shown in Fig.2.

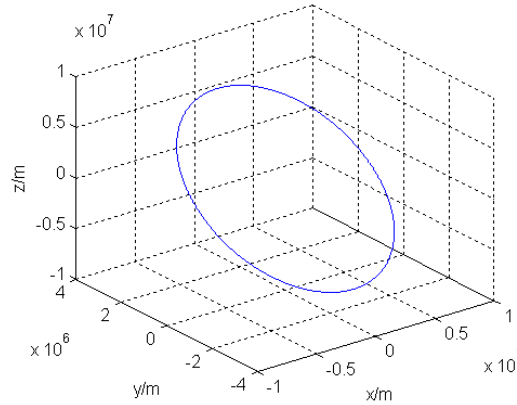


Fig.2: Nominal spacecraft orbit

#### 4.1.2. Initializations of CDKF

The initial states are

$$X_0 = [7.1365 \times 10^6 \text{ m}, 0, 0,$$

$$0, 3.1585 \times 10^3 \text{ m/s}, 6.7734 \times 10^3 \text{ m/s}]^T$$

Then the estimated initial values are

$$\hat{X}_{0|0} = X_0 + [0.1 \times 10^6 \text{ m}, 0.1 \times 10^6 \text{ m}, 0.1 \times 10^6 \text{ m},$$

$$0.1 \times 10^3 \text{ m/s}, 0.1 \times 10^3 \text{ m/s}, 0.1 \times 10^3 \text{ m/s}]^T$$

Suppose both the system noises and measurement noises are Gaussian distribution, with the covariance respectively as

$$Q = \text{diag} [q_1^2, q_1^2, q_1^2, q_2^2, q_2^2, q_2^2]$$

$$R = (3.4907 \times 10^{-4})^2$$

where  $q_1 = 1 \times 10^{-4} \text{ m}$ ,  $q_2 = 2 \times 10^{-2} \text{ m/s}$

The precisions of the star sensor and the horizon sensor are  $3''$  ( $1\sigma$ ) and  $0.02^\circ$  ( $1\sigma$ ). Three orbits are employed for the navigation simulation, and the temporary disturbances occur around 1 hour later since the navigation starts.

#### 4.2. Simulation Results

At the given simulation parameters and temporary disturbance conditions, both UKF and CDKF are employed in navigation and the errors of position and velocity are shown in Fig. 3(a) and Fig.3 (b) respectively.

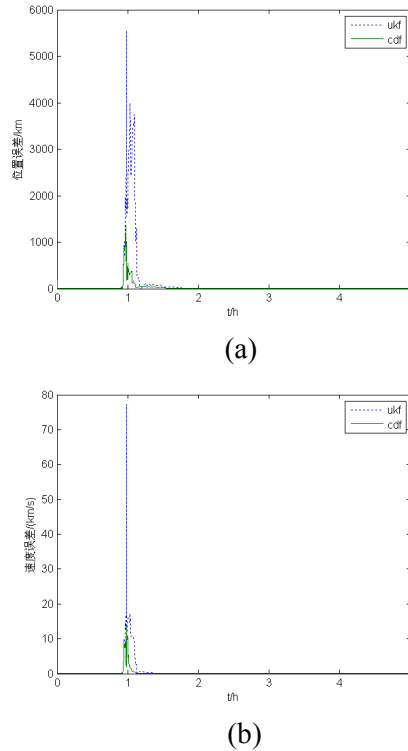


Fig.3: Errors of navigation filters: (a) Position errors; (b) Velocity errors

And the stable filter errors of both UKF and CDKF without the disturbances are also provided in TABLE 1.

TABLE 1: Errors of ukf and cdkf without disturbance

| Filter method | position error/m | velocity error /(m/s) |
|---------------|------------------|-----------------------|
| UKF           | 288.7345         | 0.275486              |
| CDKF          | 288.7598         | 0.275507              |

It is seen from TABLE 1 that, UKF and CDKF have the similar stable errors for both position and velocity. However, when disturbances exist, the estimated states by UKF have large errors, as Fig.3shows. CDKF has obvious better performances at both accuracy and the convergence speed, the error peak value is below 30% of that of UKF, and therefore has better robustness.

The reason is analyzed as below. Both UKF and CDKF can be developed from the Gaussian filter, the quadrature formula of UKF applies the exact rules of the monomial, while CDKF applies the approximate rule, which make the integrand is represented as the multiply of two separated functions. Rather, the monomial applies the unchanged formulas without considering the integrand characteristics. Moreover, although the higher order expands are both removed in calculating the posterior covariance, CDKF has less absolute error of the same order term as UKF. Therefore, although CDKF has similar accuracy as UKF, it still has better performance.

## 5. Conclusions

As the wrong state covariance will reduce the UKF estimated accuracy and the convergence rapidity, CDKF is employed in the celestial navigation of spacecraft. At the condition that the system has temporary disturbances and increased state covariance, both UKF and CDKF are applied in the navigation. The results show that the performance by CDKF is clearly superior to that by UKF, with the error peak value only 30% of that of UKF and also faster rapidity of convergence. The further analysis also shows that CDKF has low sensitivity to state covariance and therefore has better robustness than UKF in the application of celestial navigation with disturbances. However, if the disturbing environment is more complex, only CDKF is also hard to assure the navigation accuracy, the filters for stochastic jump systems should be combined for further research.

## 6. Acknowledgment

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