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Stability Of An N-Patch Predator-Prey System With Stochastic Perturbation

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Abstract. This paper discusses a randomized predator-prey dynamical system which is composed of several patches connected by linear diffusion. We show that the positive solution of the associated stochastic differential equation does not explode to infinity in a finite time. In addition, we show the positive equilibrium is global-stability under simple assumptions.

Keywords: Linear diffusion, Ito formula, Explodes, Global stability.

1. Introduction

Recently, many authors consider the effect of spatial distributions of species over the range of the habitat in population dynamics. Allen[1],by using a comparison theorem, obtained a partial answer to the persistent and extinction problem for single-species discrete diffusion systems. Kuang[2]discussed predator-prey dynamics in models of prey dispersal in two-path environments. Lu[3]discussed global asymptotic behavior in single-species discrete diffusion systems.

Li[4] considered the following predator-prey system :

$$\begin{cases} \dot{x}_i = x_i(r_i - d_i x_i - e_i y_i) + \sum_{j=1}^{n} \alpha_{ij}(x_j - x_i), \\ \dot{y} = y_i(-\gamma_i - \delta_i y_i + \varepsilon_i x_i) + \sum_{j=1}^{n} \beta_{ij}(y_j - y_i), i = 1 \cdots n. \end{cases} \tag{1}$$

Here x_i , y_i denote the densities of preys and predators on the patch i , α_{ij} , β_{ij} is the dispersal rate of preys and predators from patch j to patch i .The parameters in the system are nonnegative.

On the other hand, special population systems are often subject to environmental noise. Suppose that r_i , γ_i are stochastically perturbed, with $r_i \to r_i + (x_i - x_i^*)\dot{W}_i(t)$, $\gamma_i \to \gamma_i + (y_i - y_i^*)\dot{W}_i(t)$ .Where $\dot{W}_i(t)$ are independent white noises with $W_i(0) = 0$, $T \geq 0$ .Then this environmentally perturbed system may be described by the Ito equation

$$\begin{cases} \dot{x}_i = x_i(r_i - d_i x_i - e_i y_i) + \sum_{j=1}^{n} \alpha_{ij}(x_j - x_i) + x_i x_i^*(x_i - x_i^*)\dot{W}_i(t), \\ \dot{y} = y_i(-\gamma_i - \delta_i y_i + \varepsilon_i x_i) + \sum_{j=1}^{n} \beta_{ij}(y_j - y_i) + y_i y_i^*(y_i - y_i^*)\dot{W}_i(t), i = 1 \cdots n. \end{cases} \tag{2}$$

The parameters in the system are nonnegative. Our theory shows that the solution of Eq (2) is not only positive but also will not explode to infinity at any finite time, see Sections II. We show that the global asymptotic stability of a positive equilibrium if following conditions hold in III:

(A1) system (1.1) has a positive equilibrium $(x_1^*, y_1^*, x_2^*, y_2^*, \cdots, x_n^*, y_n^*)$;

(A2) $d_i > \frac{1}{2} x_i^{*2}$ and $\delta_i > \frac{1}{2} y_i^{*2}$;

(A3) $\alpha_{ij}\varepsilon_i x_j^* = \lambda\beta_{ij} e_i y_j^*$.

2. Positive and global solutions

Throughout this paper, $W_i(t)$ denote the independent standard Brownian motions defined on this probability space,, We introduce the notation $R_+^{2n} = \{x, y \in R_+^n\}$.

Theorem 2.1. Let the initial data $R_+^{2n}(0)$, then there exists a unique solution (x, y) to Eq. (2) on $t \geq 0$ and the solution will remain in R_+^{2n} with probability 1 under (A1) (A2).

Proof. Since the coefficients of the equation are locally Lipschitz continuous, for any given the initial data $(x(0), y(0)) \in R_+^{2n}$, where τ_e is the explosion time. To show this Solution is global; we need to show that $\tau_e = \infty$ a.s. Let be sufficiently large for every component of $\{(x,y) \in R^n : x_i > 0, y_i > 0\}$ lying within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in (0, \tau_e) : x_i(t) \notin (\frac{1}{k}, k), y_i(t) \notin (\frac{1}{k}, k), for some\ i\}$$

Where throughout this paper we set $\inf \Phi = \infty$ .Clearly, τ_k is increasing as $k \to \infty$,Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_\infty = \infty$ a.s. and $x(t) \in R_+^n$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. For if this statement is false, then there is a pair of constants $T > 0$ and $\varepsilon \in (0,1)$ Such that $P(\tau_\infty \leq T) > \varepsilon$ Hence there is an integer $k_1 \geq k_0$ such that

$$P(\tau_\infty \leq T) \geq \varepsilon \ for\ all\ k > k_1 \tag{3}$$

Define a C^2 function $V : R_+^{2n} \to R_+$ by $V(x, y) = \sum_{i=1}^{n} [\varepsilon_i(x_i - 1 - \log(x_i)) + e_i(y_i - 1 - \log(y_i))]$

The nonnegative of this function can be seen from $u - 1 - \log(u) \geq 0$ on $u > 0$ .If $(x(t), y(t)) \in R_+^{2n}$, we show that

$$d[V(x(t), y(t))] = F(x, y)dt + \sum_{i=1}^{n} \varepsilon_i x_i^*(x_i - 1)(x_i - x_i^*)dW_i(t) + \sum_{i=1}^{n} e_i y_i^*(y_i - 1)(y_i - y_i^*)dW_i(t) \tag{4}$$

Where

$$\begin{aligned} F(x, y) = &\sum_{i=1}^{n} \varepsilon_i[(x_i - 1)(r_i - d_i x_i - e_i y_i + \sum_{j=1}^{n} \alpha_{ij}(\frac{x_j}{x_i} - 1))] + \sum_{i=1}^{n} \varepsilon_i x_i^{*2}[\frac{1}{2}(x_i - x_i^*)^2] \\ &+ \sum_{i=1}^{n} e_i[(y_i - 1)(-\gamma_i - \delta_i y_i + \varepsilon_i x_i + \sum_{j=1}^{n} \beta_{ij}(\frac{y_j}{y_i} - 1))] + \sum_{i=1}^{n} e_i y_i^{*2}[\frac{1}{2}(y_i - y_i^*)^2] \\ \leq &K \end{aligned}$$

$$E(V(x(\tau_k \wedge T), y(\tau_k \wedge T))) \leq V(x(0), y(0)) + KE(\tau_k \wedge T)$$

Here, and in the sequel, $E(f)$ shall mean the mathematical expectation of f.

$$E(V(x(\tau_k \wedge T), y(\tau_k \wedge T))) \le V(x(0), y(0)) + KT$$

Set $\Omega_k = \{\tau_k \le T\}$ for $k \ge k_1$ and, by (3), $P(\Omega_k) \ge \varepsilon$ Note that for every $\omega \in \Omega_k$ there is some i such that $x_i(\tau_k, \omega)$ equals either k or $1/k$ and hence $V(x(\tau_k, \omega), y(\tau_k, \omega))$ is no less than either $k-1-\log(k)$ or $1/k-1-\log(1/k)$.

Consequently, $V(x(\tau_k, \omega), y(\tau_k, \omega)) \ge [k-1-\log(k)] \wedge [1/k-1-\log(1/k)]$

It then follows from (4) that

$$V(x(0), y(0)) + KT \ge E[1_{\Omega_k} V(x(\tau_k, \omega), y(\tau_k, \omega))]$$
$$\ge \varepsilon([k-1-\log(k)] \wedge [1/k-1-\log(1/k)])$$

Where 1_{Ω_k} is the indicator function of Ω_k .Letting $k \to \infty$ leads to the contradiction $\infty > V(x(0), y(0)) + KT = \infty$ So we have $\tau_\infty = \infty$ a.s.

3. Global stability

In this section, we will prove system (2) is global asymptotic stability under some assumptions.

Theorem 3.1 Assume (A1),(A2), (A3) hold, then the trivial solution of (2) is stochastically stable and stochastically asymptotically stable.

Proof. If $(x_1^*, y_1^*, x_2^*, y_2^*, \cdots, x_n^*, y_n^*)$ is a positive equilibrium of(1),it also is a positive equilibrium of(2).Define

$$V_i(x_i, y_i) = \varepsilon_i(x_i - x_i^* + x_i^* \ln \frac{x_i}{x_i^*}) + e_i(y_i - y_i^* + y_i^* \ln \frac{y_i}{y_i^*}) \tag{5}$$

By the Ito formula, we have

$$dV_i(x_i, y_i) = LV_i(x_i, y_i)dt + \varepsilon_i x_i^*(x_i - x_i^*)^2 dW_i(t) + e_i y_i^*(y_i - y_i^*)^2 dW_i(t)$$

Where $LV : R_+^n \times R_+^n \to R$ is defined by

$$\begin{aligned}
LV_i(x_i, y_i) &= \varepsilon_i(x_i - x_i^*)(r_i - d_i x_i - e_i y_i + \sum_{j=1}^n \alpha_{ij} \frac{x_j}{x_i} - \sum_{j=1}^n \alpha_{ij}) \\
&+ e_i(y_i - y_i^*)(-\gamma_i - \delta_i y_i + \varepsilon_i x_i + \sum_{j=1}^n \beta_{ij} \frac{y_j}{y_i} - \sum_{j=1}^n \beta_{ij}) \\
&+ \frac{1}{2}\varepsilon_i x_i^{*2}(x_i - x_i^*)^2 + \frac{1}{2} e_i y_i^{*2}(y_i - y_i^*)^2 \\
&= -\varepsilon_i(d_i - \frac{1}{2}x_i^{*2})(x_i - x_i^*)^2 - e_i(\delta_i - \frac{1}{2}y_i^{*2})(y_i - y_i^*)^2 \\
&+ \sum_{j=1}^n \alpha_{ij}\varepsilon_i x_j^*(\frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_j x_i^*}{x_j^* x_i}) + \sum_{j=1}^n \beta_{ij} e_i y_j^*(\frac{y_j}{y_j^*} - \frac{y_i}{y_i^*} + 1 - \frac{y_j y_i^*}{y_j^* y_i})
\end{aligned}$$

$$\begin{aligned}
LV_i(x_i, y_i) &\le \sum_{j=1}^n \alpha_{ij}\varepsilon_i x_j^*(1 - \frac{x_j x_i^*}{x_j^* x_i} + \ln \frac{x_j x_i^*}{x_j^* x_i}) + \sum_{j=1}^n \alpha_{ij}\varepsilon_i x_j^*(\frac{x_j}{x_j^*} + \ln \frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + \ln \frac{x_i}{x_i^*}) \\
&+ \sum_{j=1}^n \beta_{ij} e_i y_j^*(1 - \frac{y_j y_i^*}{y_j^* y_i} + \ln \frac{y_j y_i^*}{y_j^* y_i}) + \sum_{j=1}^n \beta_{ij} e_i y_j^*(\frac{y_j}{y_j^*} + \ln \frac{y_j}{y_j^*} - \frac{y_i}{y_i^*} + \ln \frac{y_i}{y_i^*}) \\
&\le \sum_{j=1}^n \alpha_{ij}\varepsilon_i x_j^*(\frac{x_j}{x_j^*} + \ln \frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + \ln \frac{x_i}{x_i^*}) + \sum_{j=1}^n \beta_{ij} e_i y_j^*(\frac{y_j}{y_j^*} + \ln \frac{y_j}{y_j^*} - \frac{y_i}{y_i^*} + \ln \frac{y_i}{y_i^*}) \\
&= \sum_{j=1}^n \alpha_{ij}\varepsilon_i x_j^*[(\frac{x_j}{x_j^*} + \ln \frac{x_j}{x_j^*} + \lambda \frac{y_j}{y_j^*} + \lambda \ln \frac{y_j}{y_j^*}) - (\frac{x_i}{x_i^*} + \ln \frac{x_i}{x_i^*} + \lambda \frac{y_i}{y_i^*} + \lambda \ln \frac{y_i}{y_i^*})] \\
&= \sum_{j=1}^n \alpha_{ij}\varepsilon_i x_j^*[G_j(x_j, y_j) - G_i(x_i, y_i)]
\end{aligned}$$

and $G_i(x_i, y_i)$ and α_{ij}, β_{ij} satisfy the assumption of reference[5],then $LV_i(x_i, y_i) \le 0$ .Therefore

$$LV(x, y) = \sum_{i=1}^{n} LV_i(x_i, y_i) \le 0$$

Especially, $LV(x, y) = 0$ if and only if $x_i = x_i^*, y_i = y_i^*$, which means the solution of (2) is stochastically stable and stochastically asymptotically stable, see [6].

4. Summary

In this paper, we show the positive solution exists of randomized predator-prey dynamical systems which are composed of several patches connected by linear diffusion. In addition, we show that the positive equilibrium is global-stability under simple assumptions.

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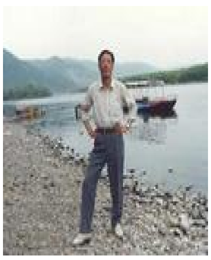

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