2010 3rd International Conference on Computer and Electrical Engineering (ICCEE 2010)
IPCSIT vol. 53 (2012) © (2012) IACSIT Press, Singapore
DOI: 10.7763/IPCSIT.2012.V53.No.2.21

The Method to Determine Spherical Harmonic Model of Asteroid based on Polyhedron

Zhang Zhenjiang[1+], Yu Meng[1], Cui Hutao[1] and Cui Pingyuan[2]

[1]Deep Space Exploration Research Center, Harbin Institute of Technology

Harbin, China

[2]School of Aerospace Science and Engineering, Beijing Institute of Technology

Beijing, China

Abstract—In order to obtain sufficiently accurate spherical harmonic model of asteroid's gravity field during the design stage of asteroid exploration mission which we can benefit from to design the orbit trajectory or the landing trajectory, the author of this paper proposed a method to determine the spherical harmonic model of asteroid's gravity field based on polyhedron model. Firstly, this method reconstructed the distributing conditions of outer gravity filed of asteroid using polyhedron model, and then chose the gravitational potential of some specific point as virtual observation. Secondly, according to the definition of the spherical harmonic model, the relationship between virtual observation and spherical harmonic coefficients can be determined. Finally, as this relationship is given, the over-determined equations can be solved to obtain the spherical harmonic coefficients in different orders of asteroid's gravity field. Comparing to the traditional method which is approximating the asteroid to a triaxial ellipsoid to determining the spherical harmonic model, the method of this paper can substantially increase the accuracy of gravity field model. Through the comparison between this paper and the spherical harmonic coefficients of asteroid 433Eros which is solved by NEAR probe's orbital data we can see that the maximum error of this paper's result is 6% at most. Whereas the maximum error of triaxial ellipsoid model can be 29.61%, which can fully indicate that the method we use has the higher accuracy which can be used to provide more accurate gravity filed model to the orbit design of the pre-mission of asteroid detection.

Keywords-Modeling of Gravity; spherical harmonic model; polyhedron model ; spherical harmonic coefficients

1. Introduction

In recent years, the detecting mission of asteroid and comet has been paid more and more attentions. The space missions which have been accomplished or in the designing process promote our exploration about a brand new field of celestial mechanics: The orbital dynamic of satellite orbiting around irregularly shaped asteroid, which is an extremely challenging problem attracting more and more scientists to do researches about it [1,2]. Thus the modeling of gravity field of irregularly shaped asteroid becomes the primary problem needed to solve to study the properties of orbital dynamics of satellite orbiting the asteroid. Using traditional method is very hard to model such complicated shaped asteroid, thus a more developed method must be introduced to solve the modeling. With the develop of computer science, especially the advancements of computer 3Dmodeling and simulating technology in the recent 20 years, the high-accurate modeling of gravity field of asteroid has becoming possible[3]. There are lots of methods of modeling the asteroid's gravity field; these methods can be classified into two categories: numerical method and analytical method [4].

[+] Corresponding author.
E-mail address: river18202@gmail.com

The main idea of numerical method is to use a polyhedron or several special shaped elements (such as balls or cubes) to approach the shape of asteroid, and then use the integral transformation method to transform triple integral of gravitational potential into curve and surface integrals which can be calculated. This method fully considered the asteroid's irregular shape, and also, since it took advances of ground astronomical observations and image information of asteroid shot by fly-by mission, the accuracy could be relatively high. Nevertheless, numerical method doesn't have analytical form, which means it can only give the gravity potential energy and the value of gravitational acceleration of given positions, the perturbing term and effect of which on the satellite's orbit cannot be analyzed either. Therefore, numerical method can only be applied in numerical simulation but not orbit design.

The main idea of analytical method is to use series expansion to approach the gravitational potential. This method has the form of analytical expression, the algorithm of which is simple and the perturbations in every order are pretty obvious. Therefore analytical method is extensively applied in various aspects of orbital dynamics and celestial mechanics, especially when we are dealing with the analysis of perturbation's effect on orbit and design of some mission obit, this method is our only choice. The biggest drawback of analytical method is that the series items' coefficients are hard to determine. For example, in spherical harmonic modeling which is mostly extensive applied, spherical harmonic coefficients C_{nm} and S_{nm} are both solved by navigation algorithm based on the data of satellite's orbit "afterward" [5]. To the asteroid which hasn't been arranged a fly-by mission yet, the accurate spherical harmonic coefficients cannot be obtained due to the lack of orbit data as observations. In the past, scholars' way to deal this is to simplify the asteroid as a triaxial ellipsoid, and then calculated the spherical harmonic coefficients in every order [6, 7]. But it is obviously inaccurate to simplify such irregularly-shaped asteroid as sphere or traxial ellipsoid.

Due to the low accuracy problem of using traxial ellipsoid to model the gravity field of asteroid in the pre-mission, and the inability of obtaining the high-accurate spherical harmonic coefficients without the satellite's fly-by data, this paper proposed a method to solve the spherical harmonic coefficients of irregularly-shaped asteroid's gravity field based on the polyhedron model. The main idea of which is firstly using the polyhedron model method to reconstruct the conditions of gravitational potential of outer gravity field of asteroid, and consider this as the virtual observations to solve the spherical harmonic coefficients of gravity field, and then according to the relationship between spherical harmonic coefficients and gravitational potential, least square method is applied to solve the spherical harmonic coefficients.

2. Method of polyhedron modeling

2.1. polyhedron model of asteroids

In 1996, R. A. Werner proposed a method which is using polyhedron to approach the asteroid's shape to solve its gravitational potential [8,9], namely the polyhedron model method, which is mostly extensive applied. As showed in figure.1, polyhedron model is a polyhedron whose surface is structured by series of triangles. Theoretically, as long as there is enough faces, polyhedron can approach any asteroid with random shape. In this way if only the distribution of gravitational potential of polyhedron model is solved can the distributing conditions of asteroid's gravitational potential be learnt.

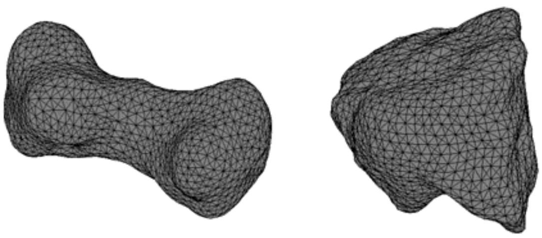

Fig.1. Polyhedron model of Asteroid 216 Kleopatra and 6489 Golevka

2.2. The computational method of gravitational potential

Using the polyhedron model method, the gravitational potential of Central objects with random shape can be defined as [10].

$$
\begin{aligned}
V(\mathbf{r}) &= -G\iiint_M \frac{1}{r}\,dm = -G\rho\iiint_V \frac{1}{r}dV \\
&= -\frac{1}{2}G\rho\iiint_V div\,\hat{\mathbf{r}}\,dV = -\frac{1}{2}G\rho\oiint_s \hat{\mathbf{n}}\bullet\hat{\mathbf{r}}dS \\
&= -\frac{1}{2}G\rho\sum_{e\in edge}\mathbf{r}_e^T\mathbf{E}_e\mathbf{r}_e\cdot L_e + \frac{1}{2}G\rho\sum_{f\in face}\mathbf{r}_f^T\mathbf{F}_f\mathbf{r}_f\cdot\omega_f
\end{aligned}
\tag{1}
$$

where **r** states the position vector of field point in body fixed frame of asteroid.

ρ is the asteroid's density.

$\hat{\mathbf{r}}$ is the unit vector of **r**, div $\hat{\mathbf{r}}$ is the divergence of **r**.

$\hat{\mathbf{n}}$ is the normal vector of infinitesimal surface dS.

$\mathbf{r}_e$ is the vector from field point to any random point on edge *e*.

$\mathbf{E}_e = \hat{\mathbf{n}}_A(\hat{\mathbf{n}}_{12}^A)^T + \hat{\mathbf{n}}_B(\hat{\mathbf{n}}_{21}^B)^T$, where $\hat{\mathbf{n}}_A$ is the normal vector of face *A*, $\hat{\mathbf{n}}_{12}^A$ is normal vector of edge *e* in the face *A*. $\hat{\mathbf{n}}_B$ and $\hat{\mathbf{n}}_{12}^B$ are similar above in face *B*, $\mathbf{E}_e$ is a 3×3 matrix .

$L_e = \ln\frac{r_{e1}+r_{e2}+e_{12}}{r_{e1}+r_{e2}-e_{12}}$, where r_{e1}, r_{e2} are the distances between field point and two endpoints on the edge, e_{12} is the length of edge *e*.

$\mathbf{r}_f$ is the vector from field point to any random point in face *f*.

$\mathbf{F}_f = \hat{\mathbf{n}}_f\hat{\mathbf{n}}_f^T$ is the face's normal vector on the face *f*, $\mathbf{F}_f$ is a 3×3 matrix.

$$\omega_f = \arctan\frac{\mathbf{r}_1\bullet(\mathbf{r}_2\times\mathbf{r}_3)}{r_1r_2r_3 + r_1(\mathbf{r}_2\bullet\mathbf{r}_3) + r_2(\mathbf{r}_2\bullet\mathbf{r}_1) + r_3(\mathbf{r}_1\bullet\mathbf{r}_2)}$$

These are the all equations we need in polyhedron model, once the position vector of one point in the body fixed frame of asteroid is given, the gravitational potential of this point can be calculated through equations above. In the next section we will introduce the method of considering gravitational potential as virtual observation to solve the spherical harmonic coefficients.

3. The algorithm of computing spherical harmonic coefficients

The algorithm similar to the LSM in the navigation algorithm is applied according to the relationship between gravitational potential and spherical harmonic coefficients to solve the spherical harmonic coefficients.

3.1. The relationship between gravitational potential and spherical harmonic coefficients.

The spherical harmonic function model of gravitational potential can be expressed as [4]:

$$V(\mathbf{r}) = \frac{\mu}{r}\left\{1+\sum_{n=1}^{N}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^m P_{nm}(\sin\varphi)[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda]\right\} + \eta \tag{2}$$

Where: μ is the gravitational constant of the asteroid.

r is the distance from field point to the centroid of asteroid.

a is the radius of asteroid's reference ellipsoid.

$P_{nm}(\sin\varphi)$ is the associated Legendre polynomials, when m=0, it deteriorates into Legendre polynomials.

C_{nm} and S_{nm} are the spherical harmonic coefficients.

η is the error of truncation.

In (2), μ/r represents the gravitational potential of homogeneous-sphere. The terms afterwards represent the amendments to the gravitational potential. Theoretically this infinite series can approach the gravitational potential of central object in any shape, but infinite series itself cannot be calculated, thus in actual use, high-order terms are usually ignored during the calculation according to the accuracy required in the mission. From (2) we know that the relationship between gravitational potential and spherical harmonic coefficients is linear, the coefficient terms are $P_{nm}(\sin\varphi)\cos m\lambda$ and $P_{nm}(\sin\varphi)\sin m\lambda$. In this way, as the distributing conditions of gravitational potential is already given, occurrence of spherical harmonic coefficient can be obtained through the linear relationship mentioned above.

3.2. Construction of linear equations set

For a random field point $\mathbf{R}^i = (\lambda^i, \varphi^i, \boldsymbol{R}^i)$, provided by solving the gravitational potential function U_i through the polyhedron modeling method can the equation be constructed taking C_{lm} and S_{lm} as variables:

$$\begin{aligned}&\left(\frac{\boldsymbol{a}}{\boldsymbol{R}^i}\right)\boldsymbol{P}_{10}\boldsymbol{C}_{10}+\left(\frac{\boldsymbol{a}}{\boldsymbol{R}^i}\right)\boldsymbol{P}_{11}\left(\boldsymbol{C}_{11}\cos\lambda^i+\boldsymbol{S}_{11}\sin\lambda^i\right)+\\&\cdots+\left(\frac{\boldsymbol{a}}{\boldsymbol{R}^i}\right)^N\boldsymbol{P}_{NN}\left(\boldsymbol{C}_{NN}\cos\boldsymbol{N}\lambda^i+\boldsymbol{S}_{NN}\sin\boldsymbol{N}\lambda^i\right)=\frac{\boldsymbol{R}^i}{\mu}\boldsymbol{V}^i-1\end{aligned}\tag{3}$$

where the number of variables in the equation is $N(N+2)$. Theoretically if $N(N+2)$ different field points are given in asteroid's gravity field, a linear equations set of $N(N+2)$ dimensional can be constructed, the form of which is given as follows:

$$\begin{aligned}&1+\left(\frac{\boldsymbol{a}}{\boldsymbol{R}^1}\right)\boldsymbol{P}_{10}^1\boldsymbol{C}_{10}+\left(\frac{\boldsymbol{a}}{\boldsymbol{R}^1}\right)\boldsymbol{P}_{11}^1\left(\boldsymbol{C}_{11}\cos\lambda^1+\boldsymbol{S}_{11}\sin\lambda^1\right)+\\&\cdots+\left(\frac{\boldsymbol{a}}{\boldsymbol{R}^1}\right)^N\boldsymbol{P}_{NN}^1\left(\boldsymbol{C}_{NN}\cos\boldsymbol{N}\lambda^1+\boldsymbol{S}_{NN}\sin\boldsymbol{N}\lambda^1\right)=\frac{\boldsymbol{R}^1}{\mu}\boldsymbol{V}^1\\&1+\left(\frac{\boldsymbol{a}}{\boldsymbol{R}^2}\right)\boldsymbol{P}_{10}^2\boldsymbol{C}_{10}+\left(\frac{\boldsymbol{a}}{\boldsymbol{R}^2}\right)\boldsymbol{P}_{11}^2\left(\boldsymbol{C}_{11}\cos\lambda^2+\boldsymbol{S}_{11}\sin\lambda^2\right)+\\&\cdots+\left(\frac{\boldsymbol{a}}{\boldsymbol{R}^2}\right)^N\boldsymbol{P}_{NN}^2\left(\boldsymbol{C}_{NN}\cos\boldsymbol{N}\lambda^2+\boldsymbol{S}_{NN}\sin\boldsymbol{N}\lambda^2\right)=\frac{\boldsymbol{R}^2}{\mu}\boldsymbol{V}^2\\&\qquad\qquad\vdots\\&1+\left(\frac{\boldsymbol{a}}{\boldsymbol{R}^{N(N+2)}}\right)\boldsymbol{P}_{10}^{N(N+2)}\boldsymbol{C}_{10}+\left(\frac{\boldsymbol{a}}{\boldsymbol{R}^2}\right)\boldsymbol{P}_{11}^{N(N+2)}\left(\boldsymbol{C}_{11}\cos\lambda^{N(N+2)}+\boldsymbol{S}_{11}\sin\lambda^{N(N+2)}\right)+\\&\cdots+\left(\frac{\boldsymbol{a}}{\boldsymbol{R}^{N(N+2)}}\right)^N\boldsymbol{P}_{NN}^{N(N+2)}\left(\boldsymbol{C}_{NN}\cos\boldsymbol{N}\lambda^{N(N+2)}+\boldsymbol{S}_{NN}\sin\boldsymbol{N}\lambda^{N(N+2)}\right)=\frac{\boldsymbol{R}^{N(N+2)}}{\mu}\boldsymbol{V}^{N(N+2)}\end{aligned}\tag{4}$$

As the equations set above can be expressed as the form of $A\mathbf{x}=b$, thus by solving this equation can C_{nm} and S_{nm} be obtained.

3.3. Solving method of equations set

Theoretically once the equations set is constructed, the spherical harmonic coefficients C_{nm} and S_{nm} can be obtained uniquely, but in actual practice we find that due to the existence of $(\boldsymbol{a}/\boldsymbol{R})^i\boldsymbol{P}_{nm}(\sin\varphi)$ and factors like that, the equations of different field points may be linearly correlated. Which makes the original determined equations set become underdetermined equations set. Sometimes though the equations are linearly independent, the condition number of matrix **A** is extremely large that matrix **A** becomes a pathological matrix, where the solutions of equations set would become meaningless due to its large errors.

There exists two ways to solve the problems above. The first way is finding a proper field point $\mathbf{R}^i = (\lambda^i, \varphi^i, \boldsymbol{R}^i)$ which causes the condition number of matrix **A** become 1.The second way is increasing the amount of equations as to make the equations set become over-determined equations, then solve the minimum norm of this equations set, which is the best estimations of C_{nm} and S_{nm}. In this paper, the second method is applied to carry out the calculations.

Theoretically the number of equations is more the results of which are better, but in the meantime, calculation speed will become slower. Therefore, a weigh of calculation speed and accuracy must be executed, by large amount of calculations we found that when the number of equations is selected two or three times bigger than the number of unknowns can the accuracy be guaranteed a high level. Needless to say, the accuracy of solutions is also related to the complexity of asteroid's shape. The number of equations should be added properly in order to guarantee the required high-accuracy of solutions.

4. Comparison verification

This section contains two examples: The first one applies the method used in this paper to calculate spherical harmonic function of a given triaxial ellipsoid, the result of which is used to compare with the true value to verify the validity of algorithm and relationship between accuracy and polyhedron parameters; the second example is to apply the method used in this paper to calculate the spherical harmonic coefficients of asteroid 433Eros, and then use the results of which to compare with the results obtained by traditional triaxial ellipsoid method and orbit data given by NEAR probe in order to verify the advantages of algorithm used in this paper and find out disadvantages relative to previous algorithms.

4.1. verification of triaxial ellipsoid

The ture value of homogeneous triaxial ellipsoid's spherical harmonic coefficients can be given by these analytical equations below [11]:

$$
\begin{aligned}
& S_{lm} = 0, \qquad l, m \in N \\
& C_{lm} = 0, \qquad l \in 2N+1,\ or\ m \in 2N+1 \\
& C_{lm} = \frac{3}{a^l} \frac{(l/2)!(l-m)!}{2^m (l+3)(l+1)!} (2-\delta_{0m}) \\
& \times \sum_{i=0}^{\mathrm{int}(\frac{l-m}{4})} \frac{\left(a^2-b^2\right)^{\frac{m+4i}{2}} \left[c^2 - \frac{1}{2}\left(a^2+b^2\right)\right]^{\frac{l-m-4i}{2}}}{16^i \left(\frac{l-m-4i}{2}\right)! \left(\frac{m+2i}{2}\right)! i!} \\
& for\ other\ l, m
\end{aligned}
\tag{5}
$$

where δ_{0m} is Kronecker symbol, the actual value of which is : $\delta_{0m} = \begin{cases} 0, & m=0 \\ 1, & m=1 \end{cases}$. a, b, c are the semi-major radius of triaxial ellipsoid, in this example the values are 16km 8km and 6km.

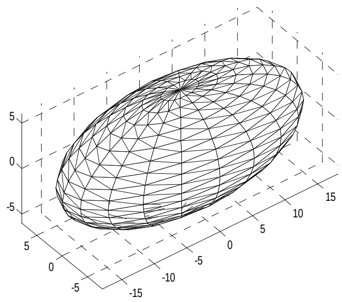

Fig.2. Polyhedral model of Triaxial Ellipsoids

Figure 2 is the polyhedron model of traixial ellipsoid used in this example. Base on the method used in this paper, the number of external surface of polyhedron is adopted by 4000 and 20000 to run the calculation. Solutions obtained are compared with the true spherical harmonic coefficients given by equation (7), the comparison results are showed in Table 1.

TABLE I. SPHERICAL HARMONIC COEFFICIENTS OF TRIAXIAL ELLIPSOID

Spherical harmonic coefficients (a = 16km)	Result		
	N=400	N=20000	
C_{10}	0	0	0
C_{11}	0	0	0
S_{11}	0	0	0
C_{20}	-0.042241	-0.043284	-0.043324
C_{21}	0	0	0
S_{21}	0	0	0
C_{22}	0.056912	0.058047	0.058095
S_{22}	0	0	0
C_{30}	0.000006	0.000006	0
C_{31}	0	0	0
S_{31}	0	0	0
C_{32}	0	0	0
S_{32}	0	0	0
C_{33}	0	0	0
S_{33}	0	0	0
C_{40}	0.008291	0.008685	0.008712
C_{41}	0	0	0
S_{41}	0	0	0
C_{42}	-0.011080	-0.011579	-0.011604
S_{42}	0	0	0
C_{43}	0	0	0
S_{43}	0	0	0
C_{44}	0.011406	0.0118641	0.011885
S_{44}	0	0	0

From table 1 we can see that:

- The method proposed by this paper can accurately solve the spherical harmonic coefficients of homogeneous triaxial ellipsoid, and difference between the given result and true value (obtained by analytical computation). As we can see in Table 1, when the face number of polyhedron is adopted as 20000, the maximum error decreases to 0.0923%.
- The error of this method decreases with the increasing of the number of outer face of polyhedron.

4.2. 433Eros example verification

Take Asteroid 433Eros as an example, in 1998. 12. 24 NEAR probe flew over 433Eros from 4100 km far away at speed 1km/s, in its fly-by mission, the size, shape and existence of magnetic field were measured. During the mission the multi-color camera on the NEAR probe took 222 photographs of 433Eros, the resolving capability of which can be limited within 500m. Based on these data mentioned above, astronomers built a 3-D model of 433Eros; the detailed data can be found at http://www.psi.edu/pds /resource/nearmod.html. The model used in this paper contains 64800 data points. Form of each point is the latitude and longitude of one single point on the surface of asteroid. Transform these data points from spherical coordinate frame to Cratesian coordinate frame, and then the polyhedron model of asteroid is obtained after triangulations [12]. As is shown in figure 3:

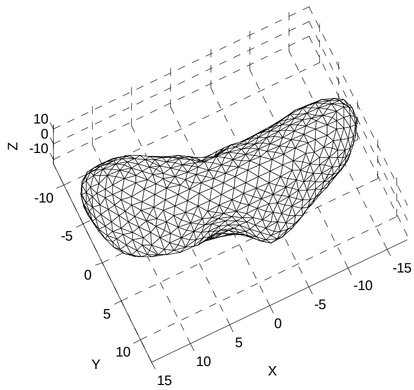

Fig.3. Polyhedron model of Asteroid Eros433

In the computation, traditional method is introduced in the first place to simplify 433Eros to a triaxial ellipsoid. The length of its three axles is 16km, 8km and 6km, after bringing these three parameters into equation (7) , the spherical harmonic coefficients can be solved, then solve the spherical harmonic coefficients again by the method used in this paper. Finally compare these two results with the result obtained by the orbit data which is given by NEAR probe, the comparison result is showed in Table 2.

As is shown in Table 2:

Comparing with the triaxial ellipsoid method, the method used in this paper can substantially increase the accuracy of results, for example, the error between C_{20} calculated by triaxial ellipsoid method and the true value (given by orbit data) is 17.44%, whereas the error between method used in this paper and true value decreases to 0.23%.

TABLE II. GRAVITY SPHERICAL HARMONIC COEFFICIENTS OF EROS433

Spherical harmonic coefficients (a = 16km)	Result		
	Triaxial ellipsoid method	Polyhedron model method	True value by NEAR
C_{10}	0	0	0
C_{11}	0	0	0
S_{11}	0	0	0
C_{20}	-0.043324	-0.052357	-0.052478
C_{21}	0	-0.000036	0
S_{21}	0	-0.000069	0
C_{22}	0.058095	0.083125	0.082538
S_{22}	0	-0.026080	-0.027745

C_{30}	0	-0.001215	-0.001400
C_{31}	0	0.004040	0.004055
S_{31}	0	0.002353	0.003379
C_{32}	0	0.001458	0.001792
S_{32}	0	-0.000593	-0.000686
C_{33}	0	-0.009789	-0.010337
S_{33}	0	-0.012287	-0.012134
C_{40}	0.008712	0.012542	0.012900
C_{41}	0	-0.000099	-0.000106
S_{41}	0	0.000158	0.000136
C_{42}	-0.011604	-0.018089	-0.017495
S_{42}	0	0.004119	0.004542
C_{43}	0	-0.000339	-0.000319
S_{43}	0	-0.000588	-0.000141
C_{44}	0.011885	0.018054	0.017587
S_{44}	0	-0.008725	-0.008939

There still exists difference between results obtained by the method used in this paper and true value. For example, the error between term S_{22} solved by this paper's method and true value is 6%, other spherical harmonic coefficients also have some difference from true value, the difference mainly comes from two respects: Firstly, the polyhedron model has constant difference from the true shape of asteroid, we can add the number of outer faces of polyhedron model to reduce this difference. Secondly, when we use the polyhedron model method, we made an assumption that the asteroid is homogeneous, but the actual asteroid is definitely not homogeneous, this could also cause some difference, which cannot be eliminated, this is also the limitations in this paper.

5. Conclusion

The method this paper proposed can be applied in obtaining the high-accurate spherical harmonic coefficients of gravity field of irregularly shaped asteroid during the pre-mission, because of its taking advantages of photograph information obtained by astro-observation or fly-by mission sufficiently, the accuracy has increased substantially comparing with the trixial ellipsoid modeling method. The accuracy of computational results given by this paper increases along with the increase of the number of outer faces on the polyhedron model. The polyhedron model has constant difference from the real shape of asteroid. For that we can add the number of outer faces of polyhedron model to reduce this difference. However, for the difference caused by the uneven distribution of asteroid's density, due to the inability of modeling the distribution of density in polyhedron model, this difference cannot be reduced or eliminated through the method used in this paper.

6. References

[1] D J Scheeres. The dynamical evolution of uniformly rotating asteroids subject to YORP. Icarus, Volume 188, Issue 2, pp. 430-450, June 2007.

[2] Sharyl M Byram, Daniel J Scheeres, and Michael R Combi. Models for the Comet Dynamical Environment. JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS. Vol. 30, No. 5, September–October, 2007.

[3] Zuber M T, D E Smith, A F Cheng, J B Garvin, O Aharonson, T D Cole, P J Dunn, Y Guo, F G Lemoine, G A Neumann, D D Rowlands, and M H Torrence. The Shape of 433 Eros from the NEAR-Shoemaker Laser Rangefinder, Science, 289, pp. 2097-2100, 2000.

[4] Stefano Casotto, Susanna Musotto. Methods for Computing the Potential of an Irregular, Homogeneous, Solid Body and its Gradient. AIAA Paper 2000-4023.

[5] J K Miller, A S Konopliv, P G Antreasian,J J Bordi,S Chesley, C E Helfrich, W M Owen, T C Wang, B G Williams, D K Yeomans, and D J Scheeres. Determination of shape, gravity and rotational state of Asteroid 433 Eros .Icarus. 155pp. 3-17, 2002.

[6] D J Scheeres. Dynamics About Uniformly Rotating Tri-Axial Ellipsoids. Applications to Asteroids. Icarus 110. pp. 225-38, 1994.

[7] P D Washabaugh and D J Scheeres. Energy and Stress Distributions in Ellipsoids, Icarus 159(2). pp.314-321, 2002.

[8] Werner, R, On the Gravity Field of Irregularly Shaped Celestial Bodies, Ph.D. thesis, The University of Texas at Austin, 1996.

[9] R.A. Werner, D.J. Scheeres. Exterior Gravitation of a Polyhedron Derived and Compared with Harmonic and Mascon Gravitation Representations of Asteroid 4769 Castalia. Celestial Mechanics and Dynamical Astronomy, 65. pp.313-44, 1997.

[10] Ryan S Park, Robert A Werner, and Shyam Bhaskaran. Estimating Small-Body Gravity Field from Shape Model and Navigation Data. AIAA/AAS Astrodynamics Specialist Conference and Exhibit 18 - 21 August, 2008.

[11] Boyce W. Comment on a Formula for the Gravitational Harmonic Coefficients of a Triaxial Ellipsoid, Celestial Mechanics and Dynamical Astronomy 67. pp. 107–110, 1997.

[12] SHAO Wei, CUI Pingyuan, CUI Hutao. Physical properties calculation of small body using points triangulations. Journal of Harbin Institute of Technology. Vol.42, No.5. pp.687-691. 2010.5.