*2010 3rd International Conference on Computer and Electrical Engineering (ICCEE 2010)*
***IPCSIT vol. 53 (2012) © (2012) IACSIT Press, Singapore***
*DOI: 10.7763/IPCSIT.2012.V53.No.1.48*

# Feedback Control for Uncertain Systems with Limited Data Rates

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**Abstract.** This paper addresses the stabilization problem for stochastic uncertain systems with bounded process disturbance, where sensors, controllers and plants are connected by a digital communication channel. A lower bound on data rates of the channel, above which there exists a quantization, coding and control policy to stabilize the unstable plant in the mean square sense. A sufficient condition for stabilization is derived. Simulation results show the validity of the proposed control policy.

**Keywords:** Networked control systems; digital communication channels; communication constraints; feedback stabilization

## 1. Introduction

In modern control theory, communication is an important component of distributed and networked control systems. The main purpose in many engineering applications is to control one or more dynamical systems by employing multiple sensors and actuators communicating over a digital communication channel. In this framework, the existence of a critical positive date rate, above which there exists a quantization, coding and control policy to stabilize an unstable plant (see [6]).

Various publications in this field have introduced necessary and sufficient conditions for the stabilization of linear systems in the presence of data-rate constraints. [1-5] showed that the construction of a stabilizing controller required that the data rate of the feedback loop was above a lower bound. [6-10] investigated different notions of stability, such as moment stability, and stability in the almost sure sense. [11, 12] addressed the problem of state estimation in the presence of information constraints. The emerging area of control with limited data rates incorporates ideas from both control and information theory. There exists a critical positive data rate below which there exists no quantization and control scheme to stabilize an unstable plant (see [22-25]).

Since the sampling data and controller signals are transmitted through a network, networked-induced delays and data dropout in NCSs are always inevitable. The stability analysis and stabilization controller design for NCSs were investigated by many researchers (see [13-17]). Control under communication constraints inevitably suffers signal transmission delay, date packet dropout and measurement quantization which might be potential sources of instability and poor performance of control systems (see [20, 21]). [18] addressed the problem of robust $H_\infty$ estimation for uncertain systems subject to limited communication capacity and [19] addressed the design of robust $H_\infty$ controllers for uncertain networked control systems where both networked-induced delay and data dropout were considered.

In this technical note, we investigate the stabilization problem for stochastic uncertain time-varying systems, where the feedback loop employs a digital communication link with a stochastically time-varying

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data rate. Our main purpose here is to present a lower bound of data rates, above which there exists a quantization, coding and control policy to stabilize the unstable plant.

The remainder of this paper is organized as follows: Section 2 introduces problem formulation. Section 3 deals with the feedback stabilization problem. The results of numerical simulation are presented in Section IV. Conclusions are stated in Section V.

## 2. Problem Formulation

We address the stabilization problem for stochastic uncertain systems under communication constraints. Motivated by the type of constraints that arise in many computer networks, we consider the following class of networked control systems:

$$X(t+1)=A(1+d_e(k))X(t)+BU(t)+FW(t),$$
$$Y(t)=X(t)+V(t) \quad (1)$$

where, $X(t)\in R^n$ is the plant state, $U(t)\in R^q$ is the control input, $Y(t)\in R^n$ is the observation output, $V(t)\in R^n$ is the measurement noise, and $W(t)\in R^p$ is the process disturbance, respectively. A, B and F are known constant matrices with appropriate dimensions (see Fig.l). The initial position $X_0$ is a random vector. Without loss of generality, suppose that the pair (A, B) is controllable and the system is fully observable. $d_e(t)$ denotes uncertainty in the knowledge of A(t). We choose this structure since it allows the denotation of a wide class of model uncertainty. It also allows the construction of most stabilizing schemes.

We adopt [26] as a primary reference and give the following definitions.

Let x and y denote two random variables. We write $\log_2(.)$ simply as log(.). The differential entropy h(x) is defined as

$h(x):=E_x[\log 1/p(x)]$.

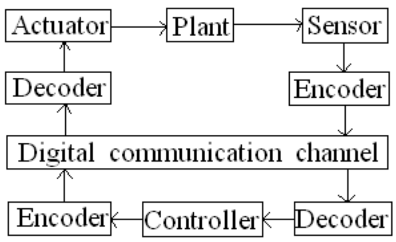


Figure 1. System with communication channel.

The conditional differential entropy of x given y is defined as

$$h(x|y):=E_{x,y}[\log 1/p(x|y)].$$

The mutual information between x and y is defined as

$$I(x;y):=h(x)-h(x|y)=E_{x,y}[\log p(x|y)/p(x)].$$

The information rate distortion function between x and y is defined as

$$R(D):=\inf_{p(y|x)\in Xi}\{ I(x;y)\}$$

with $Xi=\{p(y|x):E[d(x, y)]<D\}$. Therein, d(x, y) denotes a distortion function or distortion measure, which is a mapping $d: x\times y\rightarrow R^+$ and D is a given constant.

The information transmission rate Re (t) denotes the amount of information transmitted in unit time. Re(t) is defined as

$$Re(t)= 1/h\ I(x;y) \text{ (bits/s)}$$

where h is the transmission time for I(x;y).

Similar to that in [27], the quantization and coding scheme to be applied is presented. Given a positive integer M and a nonnegative real number De(t), we define the quantizer q: R→Z with sensitivity De(t) and saturation value M by the formula

$$q(z)=\begin{cases} M^{+}, & if \quad z>(M+\frac{1}{2})De \\ M^{-}, & if \quad z<-(M+\frac{1}{2})De \\ \left\lfloor \frac{z}{De}+\frac{1}{2} \right\rfloor, & if \quad -(M+\frac{1}{2})De<z<(M+\frac{1}{2})De \end{cases} \tag{2}$$

where define $\lfloor x \rfloor$:= max{k∈Z:=k<x, x∈R}. The indexes $M^+$ and $M^-$ will be employed if the quantizer saturates. The scheme to be used here is based on the hypothesis that it is possible to change the sensitivity (but not the saturation value) of the quantizer on the basis of available quantized measurements. The quantizer may counteract disturbances by switching repeatedly between "zooming out" and "zooming in" (see [27]).

Next, we encode $q(X_i(t))$ and compute $C_j$∈{0,1} satisfying the following equality:

$$q(X_i(t)) = \text{sum}_{j=0}^{ri(t)-1} C_j 2^j \tag{3}$$

Then we place $(C_0, C_1, \ldots, Cr_{i(t)-1})$ for transmission. Thus, the data rate R(t) is obtained by

$$R(t) = \text{sum}_{i=1}^{n} r_i(t).$$

The controller at the other end of the communication channel may obtain the estimate of $X_i(t)$ by

$$X_i(t) = De(t)\text{sum}_{i=0}^{ri(t)-1} C_j 2^j \quad (i=1,\ldots,n)$$

Our main problem here is to derive a sufficient condition for stabilization the system (1) with the quantization scheme (2) and coding scheme (3) in the mean square sense

$$\lim_{t\to\infty}\sup E||X(t)||^2<\infty \tag{4}$$

by employing limited data rates of the feedback loop.

## 3. Feedback Stabilization of Uncertain Systems

This section deals with the stabilization problem for stochastic uncertain systems with limited data rates. We will derive sufficient conditions for stabilization of the unstable plant. The proof techniques we employ come from [25]. We stress that our main work is to extend the results in [25].

Here, assume without loss of generality that $E||d_e(t)||<P_{de}$ and $E||W(t)||<P_W$. We define the upper-bound sequence as

$$p_{hi}(t+1)=|\det(A)|(1+P_{de})p_{hi}(t)+||F||P_W.$$

Then, we first give the following lemma.

**Lemma 3.1:** Consider the system (1). If assume that $E||d_e(t)||<P_{de}$ and $E||W(t)||<P_W$ and $E||X(0)||<P_0$, then the following holds:

$$E||X(t)||<p_{hi}(t) \quad \text{for all time t.}$$

**Proof:** First we assume that $E||X(t)||<p_{hi}(t)$ holds. Then
we get

$$E\|X(t+1)\|<|\det(A)|(1+E\|d_e(t)\|)E\|X(t)+A^{-1}BU(t)\|+E\|FW(t)\|.$$

Since the encoder constructs the binary expansion of the plant state by the quantization scheme (2) and coding scheme (3). Then, the way allows us to conclude that

$$E\|X(t)+A^{-1}BU(t)\|<p_{hi}(t).$$

It implies that

$$E\|X(t+1)\|<|\det(A)|(1+P_{de})p_{hi}(t)+\|F\|P_w=p_{hi}(t+1).$$

Furthermore, notice that

$$E\|X(0)\|<P_0<p_{hi}(0)$$

holds. We may obtain $E\|X(t)\|<p_{hi}(t)$. Thus, $E\|X(t)\|<p_{hi}(t)$ holds for all time t.

Then, we derive the sufficient condition for stabilization.

**Theorem 3.1:** Consider the system (1) with the quantization scheme (2) and coding scheme (3). Assume that $E\|d_e(t)\|<P_{de}$ and $E\|W(t)\|<P_W$ and $E\|X(0)\|<P_0$. Define M(t) as $M(t):=2^{t(R(t)-R\_0)}$ where

$$R_0=-\log(2^{de(t)}+p_{hi}(t)).$$

Then, the system (1) is stabilizable in the mean square sense (4) if the data rate satisfies the following inequality:

$$R(t)>\log|\det(A)|+\log M(t)\ \text{(bits/sample)}. \tag{5}$$

If (5) are satisfied, then the following holds:

$\lim_{t\to\infty}\sup E\|X(t)\|<1/2^{R(t)-R\_0\}-1}\|F\|P_W.$

**Proof:** Notice that

$$p_{hi}(t+1)=2^{t(R(t)-R\_0)}p_{hi}(t)+\|F\|P_W.$$

We solve the difference equation above for t>1 and obtain

$$p_{hi}(t)=2^{t(R(t)-R\_0)}p_{hi}(0)+\text{sum}_{i=0}^{t-1}2^{(t-i-1)(R(t)-R\_0)}\|F\|P_W.$$

Furthermore, we know that

$$\begin{aligned}h(X(t+1))&=h[A(1+d_e(t))X(t)+BU(t)+FW(t)]\\&=h[A(1+d_e(t))X(t)+FW(t)]\\&=\log|\det(A(1+d_e(t)))|+h(X(t))+h(FW(t)).\end{aligned}$$

In order to ensure stabilization of the system (1) in the mean square sense (4), we must transmit the information amount of the pant states greater than

$$\text{Log}\ |\det(A(1+de(t)))|+\log\|F\|P_W.$$

If assume that

$$R(t)>t(R(t)-R_0)+\log|\det(A)|$$

then, we have $\lim_{t\to\infty} E|p_{hi}(t)|<\infty$. By Lemma 3.1, we have

$$\text{Lim}_{t\to\infty} E\|X(t)\|<1/2^{(R(t)-R\_0)\}-1}\|F\|P_W<\infty.$$

It means that the system (1) is stabilizable in the mean square sense. Thus, we obtain

$$R(t)>\log|\det(A)|+\log M(t).$$

Next, we give a sequence $p_{him}(t)$ in the following theorem. The sequence $p_{him}(t)$ is suitable for the analysis in the presence of uncertainty, which is propagated according to a first-order difference equation.

**Theorem 3.2:** Consider the system (1) with the quantization scheme (2) and coding scheme (3). Assume that $E\|d_e(t)\|<P_{de}$ and $E\|W(t)\|<P_W$ and $E\|X(0)\|<P_0$. The following sequence is defined as

$$p_{him}(t)=h(t)p_{hi}(0)+\|F\|\ P_W \text{ sum }_{i=0}^{t-1}h(t-i-1)$$

where $p_{him}(i)=0$ for $i<0$ and $h(t)$ is the impulse response given by

$$h(t)=E[2^{m(\log(|\det(A)|-R0))t/m}$$

and the following holds:

$$E\|X(t)\|^m<p_{him}(t).$$

**Proof:** First, we assume that $p_{hi}(t)<_{phim}(t)$ holds. It follows from Lemma 3.1 that

$$E\|X(t)\|<p_{hi}(t).$$

This implies

$$E\|X(t)\|<p_{him}(t)$$

holds. Furthermore, notice that

$$E\|X(t+1)\|<p_{hi}(t+1). \tag{6}$$

Thus, we only need to derive that

$$p_{hi}(t+1)<p_{him}(t+1).$$

Then, we have

$$E[p_{hi}^m(t+1)]^{1/m}=E[(\log|\det(A)|(1+P_{de})p_{hi}(t)+\|F\|P_W)^m]^{1/m}.$$

It follows that $p_{hi}(t)$ is independent of de(t). Thus, we get

$$E[p_{hi}^m(t+1)]^{1/m}< E[\ |\det(A)|(1+P_{de})]\ E[p_{hi}^m(t)]^{1/m}+\|F\|P_W.$$

This implies

$$E[p_{hi}^m(t+1)]^{1/m}< E[2^{m(\log|\det(A)|+R0)}]^{1/m}p_{him}(t)+\ \|F\|P_W$$
$$< E[p_{him}^m(t)]^{1/m}$$

which is equivalent to

$$p_{hi}(t+1)<p_{him}(t+1).$$

The proof is complete once we substitute the inequality above to (6).

**Remark 3.2:** Theorem 3.2 states that, the system (1) may be stabilizable in the mean square sense if the data rate of the digital communication channel satisfies the condition given by Theorem 3.2. Due to the uncertainty, the data rate R (t) is time-varying and greater than log |det(A)| in order to guarantee stabilization of the system (1).

# 4. Simulations

This section presents a numerical example to illustrate the effectiveness of limited data rates for stabilization of stochastic uncertain systems. We present an open-loop unstable system as follows:

$$X(t+1)=(1+d_e)\begin{bmatrix}3.56 & 0.85\\ -0.12 & 3.82\end{bmatrix}X(t)+\begin{bmatrix}2.2\\ 1.8\end{bmatrix}U(t)+W(t).$$

Let $X(0)=[5\ \ -5]^T$. Assume that $P_{de}=1$ and $P_W=0.2$. A simulation is given in Fig. 2.

**Remark 4.1:** The obtained state responses are shown in Fig.2. It states that the coding-control scheme based on Theorem 3.1 and Theorem 3.2 can stabilize the system above.

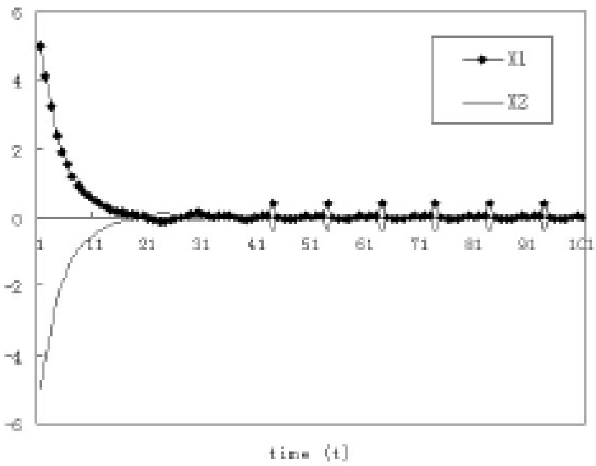


Figure 2. Trajectory of plant states.

# 5. Conclusion

In this technical note, we addressed the stabilization problem for stochastic uncertain systems where sensors, controllers and plants are connected by a digital communication channel. A sufficient condition for stabilization was derived. A lower bound of data rates of the feedback loop, above which there exists a quantization, coding and control scheme to stabilize the unstable plant in the mean square sense. The simulation results have illustrated the effectiveness of the proposed scheme.

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