

Research on Steering System's Yaw Angle Velocity of Vehicle's Two-wheel Model

Bin Zhong and Renjun Zhan

Equipment & Transportation Department; Research Institute of Non-lethal Weapons, Engineering College of Chinese Armed Police Force, Xi'an, China

Abstract—In order to study the characteristic of yaw angle velocity for two-wheel vehicle model's steering system, the linear dynamic model of vehicle's two-wheel steering system is established with dynamic research method. Conclusions were drawn: There was a critical vehicle velocity when the stability factor was negative. The steering system was steady when vehicle's velocity was less than critical vehicle velocity. The yaw angle velocity gain would be infinite when the vehicle's velocity was critical vehicle velocity. The over steering vehicle's gain curve shape was up convex and the under steering vehicle's gain curve shape was down concave. Damping ratio mainly affected dynamic response's overshoot and natural frequency mainly affected dynamic response's rise time and peak time.

Keywords-vehicle; steering system; yaw angle velocity; yaw angle velocity gain; steering dynamic response

1. Introduction

The function of vehicle's steering system is changing or keeping vehicle's driving or regressing direction [1] [2] and [3]. Dynamic analysis and modeling for steering system is important process for realizing yaw motion control whiling vehicle's steering [4], [5] and [6]. There are two degrees of freedom, lateral linear displacement and yaw angle displacement when vehicle is steering [7] and [8]. In order to study on the relationship between yaw angle velocity and steering wheel's steering angle and study on yaw angle velocity's dynamic response for steering angle input, the linear dynamic model of vehicle's two-wheel steering system is established with dynamic research method according to two-degree-of-freedom vehicle model. According to the linear dynamic model of vehicle's two-wheel steering system, we analyze the steering system's stability and three steering characteristic and research the steering system's instantaneous response for yaw angle velocity. The research method and conclusions in this paper can offer theory basis to vehicle's yaw motion control.

2. Dynamic research on steering system based on two-degree-of-freedom vehicle model

Figure 1 is showing the two-degree-of-freedom vehicle model and the steering motion's geometric relation. Moving coordinate system xoy is consolidated on the vehicle, namely vehicle coordinate system, x axis is vehicle's longitudinal axis, y axis is vehicle's lateral axis, o is the origin of coordinate, and o is reclosing on the vehicle's mass center C . O is vehicle's instantaneous centre of turn. Every point on the vehicle is instantaneously rotating around point O . Mass center C 's velocity is v_c , namely C 's tangential velocity when C is instantaneously rotating around point O . u is projection on x axis and v is C 's projection on y axis. ρ is mass center's radius of gyration. a is mass center's absolute acceleration. Vehicle

will generate lateral linear displacement y because of v . Anticlockwise angle is positive angle. φ_r is yaw angle displacement around vertical axis z and ω_r is yaw angle velocity around vertical axis z . z axis is outward and vertical to paper surface. u_f and u_r are velocity vectors of vehicle's front and rear axletree respectively. v_A and v_B are yaw motion's tangential velocity of front and rear wheel around z axis, respectively, namely rotary tangential velocity of point A and B around mass center C , and $v_A = \omega_r l_f + v$, $v_B = \omega_r l_r - v$. δ_f is front wheel's steering angle through steering wheel's input. θ is angle between front velocity vector u_f and x axis. α_f and α_r are angles between u_f and tyre plane, u_r and tyre plane, respectively, because F_{yf} and F_{yr} are cornering forces acting on front tyre A and rear tyre B . $\theta = \delta_f - \alpha_f$ according to figure 1. β is angle between vehicle's mass center velocity v_c and x axis, and β is called as vehicle's side-slip angle. Steering motion's two-degree-of-freedom is showed by lateral displacement y and yaw angle displacement φ_r . L is distance from vehicle's front axletree to rear axletree. l_f and l_r are distances from mass center C to vehicle's front axletree and from C to vehicle's rear axletree, respectively.

In order to express conveniently, firstly expressions of tyre's cornering forces F_{yf} and F_{yr} are given. The tyre's cornering force is proportional to tyre's side-slip angle because α_f and α_r are small. According to figure 1, we have the following expressions

$$\begin{cases} F_{yf} = C_{af} \alpha_f \\ F_{yr} = C_{ar} \alpha_r \end{cases} \quad (1)$$

where: C_{af} and C_{ar} are proportional coefficients, are called as cornering stiffness, and the unit of C_{af} or C_{ar} is N/rad. According to the direction relationship between tyre's cornering force and tyre's side-slip angle, the cornering stiffness is negative, namely $C_{af} < 0$ and $C_{ar} < 0$.

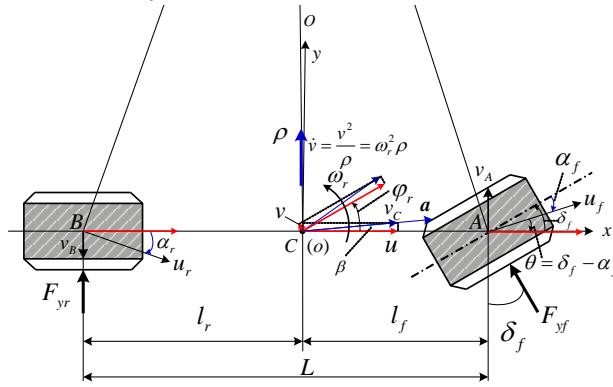


Figure1:Two-degree-of-freedom vehicle model and vehicle's steering motion

We have the following steering system's dynamic equations according to expression (1) and figure 1,

$$\begin{cases} m(\dot{v} + \omega_r u) = C_{af} \alpha_f \cos \delta_f + C_{ar} \alpha_r \\ I_z \dot{\omega}_r = C_{af} \alpha_f l_f \cos \delta_f - C_{ar} \alpha_r l_r \end{cases} \quad (2)$$

where: m is vehicle's mass; I_z is vehicle's rotational inertial around z axis, I_z 's unit is $\text{kg} \cdot \text{m}^2$.

Because tyre's side-slide angle α_f and α_r difficultly measured directly, α_f and α_r are expressed with vehicle's side-slide angle β , yaw angle velocity ω_r , mass center velocity v_c 's longitudinal projection u and front wheel's steering angle δ_f . α_f and α_r are clockwise direction from figure 1, namely $\alpha_f < 0$ and $\alpha_r < 0$. So, we have the following approximate expression, $\theta = \frac{v_A}{u}$, $\alpha_r = -\frac{v_B}{u}$, $\beta = \frac{v}{u}$ and $\dot{v} = u\dot{\beta}$ when vehicle's longitudinal velocity u is constant.

Then, we have the following expression according to $\theta = \delta_f - \alpha_f$,

$$\begin{cases} \alpha_f = \beta + \frac{\omega_r l_f}{u} - \delta_f \\ \alpha_r = \beta - \frac{\omega_r l_r}{u} \end{cases} \quad (3)$$

Take expression (3) into the expression (2), we obtain the following vehicle's two-degree-of-freedom linear steering system dynamic model based on the vehicle's lateral displacement y and yaw angle velocity ω_r ,

$$\begin{cases} m(\dot{v} + u\omega_r) = (C_{af} + C_{ar})\beta + \frac{\omega_r}{u}(C_{af}l_f - C_{ar}l_r) - C_{ar}\delta_f \\ I_z\dot{\omega}_r = (C_{af}l_f - C_{ar}l_r)\beta + \frac{\omega_r}{u}(C_{af}l_f^2 + C_{ar}l_r^2) - C_{ar}l_f\delta_f \end{cases} \quad (4)$$

We can change expression (4) to the following expression about vehicle's side-slip angle velocity $\dot{\beta}$ and yaw angle acceleration $\dot{\omega}_r$,

$$\begin{cases} \dot{\beta} = \frac{C_{af} + C_{ar}}{mu}\beta + \frac{C_{af}l_f - C_{ar}l_r - mu^2}{mu^2}\omega_r - \frac{C_{af}}{mu}\delta_f \\ \dot{\omega}_r = \frac{C_{af}l_f - C_{ar}l_r}{I_z}\beta + \frac{C_{af}l_f^2 + C_{ar}l_r^2}{I_zu}\omega_r - \frac{C_{af}l_f}{I_z}\delta_f \end{cases} \quad (5)$$

We express the expression (5) with matrix form, namely

$$p \begin{bmatrix} \beta \\ \omega_r \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \beta \\ \omega_r \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \delta_f \quad (6)$$

where: $p = \frac{d}{dt}$ is differential operator; $a_{11} = \frac{C_{af} + C_{ar}}{mu}$; $a_{12} = \frac{C_{af}l_f - C_{ar}l_r - mu^2}{mu^2}$; $a_{21} = \frac{C_{af}l_f - C_{ar}l_r}{I_z}$; $a_{22} = \frac{C_{af}l_f^2 + C_{ar}l_r^2}{I_zu}$; $b_{11} = -\frac{C_{af}}{mu}$; $b_{21} = -\frac{C_{af}l_f}{I_z}$.

Generally, the ratio of yaw angle velocity $\omega_r(t)$ and front wheel's steering angle $\delta_f(t)$ is used to evaluate steering system's stability. The expression (12) is transformed with Laplace transformation. So, we obtain the following transfer function between the yaw angle velocity and the front wheel's steering angle,

$$G(s) = \frac{\omega_r(s)}{\delta_f(s)} = \frac{b_{21}s + a_{21}b_{11} - b_{21}a_{11}}{s^2 - (a_{11} + a_{22})s - a_{12}a_{21} + a_{11}a_{22}} \quad (7)$$

3. Analysis of stability and steering characteristic for steering system

We know that the steering system is second-order system according to the expression and the characteristic equation is

$$s^2 - (a_{11} + a_{22})s - a_{12}a_{21} + a_{11}a_{22} = 0 \quad (8)$$

Take $a_{11} \sim a_{22}$ into characteristic equation, then,

$$muI_zs^2 - [m(C_{af}l_f^2 + C_{ar}l_r^2) + I_z(C_{af} + C_{ar})]s + \frac{C_{af}C_{ar}}{u}L^2(1 + Ku^2) = 0 \quad (9)$$

where: $K = \frac{m}{L^2}(\frac{l_f}{C_{ar}} - \frac{l_r}{C_{af}})$ is called as steering system's stability factor and K 's unit is s^2/m^2 . Obviously, K is

related to wheel base L , vehicle's mass m and tyre's cornering stiffness. K 's value may be positive, negative and zero. K reflects vehicle's steering characteristic itself. K is important parameter that characterizes vehicle's steady response. K may be determined after the vehicle is designed completely and K is not related to vehicle's driving condition and application condition.

Because of $C_{af} < 0$ and $C_{ar} < 0$, plural variable s^2 on degree term is positive. According to Louts criterion, the necessary and sufficient condition is that every term's coefficient is positive if the second-order linear system is stable. When the steering system is stable, the vehicle's velocity

$$u < u_{crit} \quad (10)$$

where: u_{crit} is called as critical vehicle velocity, and $u_{crit} = (-K)^{-0.5}$, $K < 0$; it must be pointed out that the critical vehicle velocity would be meaningful, only while $K < 0$. Obviously, the steering system is not stable while $u \geq u_{crit}$.

Let $s = 0$ in expression (7) and take $a_{11} \sim a_{12}$, b_{11} and b_{21} into expression (7), and the following expression is right

$$G_{\omega_r-\delta_f} = \frac{u/L}{1 + Ku^2} \quad (11)$$

$G_{\omega_r-\delta_f}$ is called as yaw angle velocity gain and it's unit is s^{-1} .

According to K 's different signal, we classify the vehicle's steering characteristic into neutral steering ($K = 0$), under steering ($K > 0$) and over steering ($K < 0$), respectively. We call these vehicles with these steering characteristic as neutral steering vehicle, under steering vehicle and over steering vehicle, respectively. $G_{\omega_r-\delta_f}$ is function about vehicle velocity u and stability factor from expression. Surface $G_{\omega_r-\delta_f}(K, u)$ is showing in (a) of figure 2 according to different K and different u , and curves $G_{\omega_r-\delta_f} - u$ are showing in (b) of figure 2. We call curve $G_{\omega_r-\delta_f} - u$ as vehicle's yaw angle velocity gain curve.

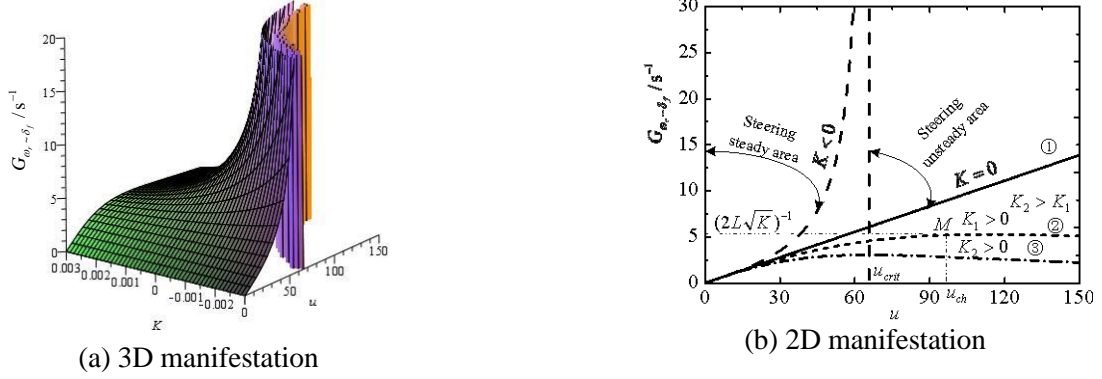


Figure 2. Three steering characteristic, steering steady area and steering unsteady area

Obviously, $G_{\omega_r-\delta_f} < u \cdot L^{-1}$ while $K > 0$, the yaw angle velocity gain is proportional to vehicle's velocity (curve ① in figure 2 (b)).

$G_{\omega_r-\delta_f} < u \cdot L^{-1}$ while $K > 0$, the curve ② and ③ of is lower than curve ①, and the shape of curve ② and ③ is up convex, there is a maximum point will lower with K 's increase, at the same time under steering quantity will increase. The vehicle's velocity is $u_{ch} = K^{-0.5}$ at the point M , and call u_{ch} as characteristic speed for the under steering vehicle and $(G_{\omega_r-\delta_f})_{max} = (2L\sqrt{K})^{-1}$. u_{ch} is an important parameter that characterizes the under steering quantity, and K will increase and u_{ch} will decrease with the under steering quantity's increase.

$G_{\omega_r-\delta_f} > u \cdot L^{-1}$ while $K < 0$, the curve ④ is higher than curve ①, and the shape of curve ④ is down concave. $u_{crit} = (-K)^{-0.5}$ will decrease when K decreases, namely $|K|$ increases and the yaw angle velocity gain curve is closer to longitudinal axis. Over steering vehicle's yaw angle velocity gain $G_{\omega_r-\delta_f} = \infty$ while $u = u_{crit}$, and vehicle's steering motion will lose stability, and only minimum front wheel's steering angle will produce infinite yaw angle velocity. So the vehicle will be suddenly turned and vehicle will sideslip and heel very possibly. The vehicles with over steering characteristic all have danger of losing stability.

On the other hand, from stability analysis for steering system, we know that the steering system is steady while $u < u_{crit}$ and u_{crit} is meaningful only while $K < 0$. So in (b) of figure 2, there are steering steady area and steering unsteady area for vehicle's steering system while $K \leq 0$.

4. Analysing on yaw angle velocity's instantaneous response for steering system

Rewrite the expression (7) as

$$G(s) = G_{\omega_r-\delta_f} \frac{\frac{\tau}{T^2}s + \frac{1}{T^2}}{s^2 + 2\frac{\xi}{T}s + \frac{1}{T^2}} \quad (12)$$

where: τ and T are steering system's time constants; ξ is called as damping ratio or relative damping factor;

$$\tau = -\frac{m u l_f}{C_{ar} L}, \quad T = \frac{u}{L \sqrt{\frac{C_{af} C_{ar}}{m I_z} (1 + K u^2)}}, \quad \xi = -\frac{m(C_{af} l_f^2 + C_{ar} l_r^2) + I_z (C_{af} + C_{ar})}{2L \sqrt{m I_z C_{af} C_{ar} (1 + K u^2)}}, \quad \text{and } \xi > 0.$$

Now, discuss yaw angle velocity's instantaneous response characteristic. The steering system's characteristic equation is

$$s^2 + 2\xi \omega_n s + \omega_n^2 = 0 \quad (13)$$

where: $\omega_n = \frac{1}{T}$ is called as natural frequency or no damped oscillation frequency.

$\omega_r(s)$ is multiplied in expression (13), and expression (13) is transformed with inverse Laplace. So we have the following expression,

$$\ddot{\omega}_r(t) + 2\xi\omega_n\dot{\omega}_r(t) + \omega_n^2\omega_r(t) = 0 \quad (14)$$

Let yaw angle velocity's initial value be $\omega_r(0) = \omega_{r0}$, then, $\omega_r(t)$'s solution is

$$\omega_r(t) = \begin{cases} \frac{\omega_{r0}}{2(\xi^2 - 1)} [(\xi^2 - 1 + \xi\sqrt{1 - \xi^2})e^{-(\xi - \sqrt{1 - \xi^2})\omega_n t} + (\xi^2 - 1 - \xi\sqrt{1 - \xi^2})e^{-(\xi + \sqrt{1 - \xi^2})\omega_n t}] & (0 < \xi < 1) \\ \omega_{r0}(1 + \omega_n) + e^{-\omega_n t} & (\xi = 1) \\ \frac{\omega_{r0}}{2(\xi^2 - 1)} [(\xi^2 - 1 + \xi\sqrt{\xi^2 - 1})e^{-(\xi - \sqrt{\xi^2 - 1})\omega_n t} + (\xi^2 - 1 - \xi\sqrt{\xi^2 - 1})e^{-(\xi + \sqrt{\xi^2 - 1})\omega_n t}] & (\xi > 1) \end{cases} \quad (15)$$

According to $0 < \xi < 1$, $\xi = 1$ and $\xi > 1$, we call the yaw angle velocity's time domain response as under damping instantaneous response, critical damping instantaneous response and over damping instantaneous response, respectively.

Generally, vehicle's steering instantaneous responses are all under damping instantaneous response. So, we only discuss yaw angle velocity's change laws while $0 < \xi < 1$ when the steering wheel's input is angle step input (namely, steering wheel's steering angle is suddenly increased to a fixed value from zero). According to expression (12), the following expression is right

$$\omega_r(s) = G_{\omega_r - \delta_f} \frac{\delta_0}{s} \cdot \frac{\tau + 1}{T^2 s^2 + 2\xi T s + 1} \quad (16)$$

where: δ_0 is step function's amplitude.

After expression (16) is transformed with inverse Laplace, we obtain yaw angle velocity step response's time domain function

$$\omega_r'(t) = \frac{u/L}{1 + Ku^2} \delta_0 \left\{ 1 - \left[\frac{\omega_n \tau - \xi}{\sqrt{\xi^2 - 1}} \sinh(\omega_n \sqrt{\xi^2 - 1} t) + \cosh(\omega_n \sqrt{\xi^2 - 1} t) \right] e^{-\xi \omega_n t} \right\} \quad (17)$$

5. Simulation example

Now simulation parameters are set: $m = 1640\text{kg}$, $l_f = 1.105\text{m}$, $l_r = 1.345\text{m}$, $C_{af} = -33020\text{N/rad}$, $C_{ar} = -55830\text{N/rad}$, $I_z = 2720\text{kg}\cdot\text{m}^2$, $u = 20\text{m/s}$. $K = 0.0057\text{s}^2/\text{m}^2$ after calculating.

According to expression (9), if $1 + Ku^2 > 0$ is tenable, the steering system would be stable. Yaw angle velocity's steering angle impulse response, angle steep response, ramp response are showing in figure 4 when vehicle's velocity is $20\text{m}\cdot\text{s}^{-1}$, $40\text{m}\cdot\text{s}^{-1}$ and $60\text{m}\cdot\text{s}^{-1}$, respectively. Especially, vehicle's steady response is uniform circular motion when $\delta_f(t)$ is angle step input, and this steady response is called as steering angle step input's steady response. We know about from figure 3: the yaw angle velocity will attain some steady value and the vehicle's centripetal acceleration $\dot{v} = 0$ after some time. So the steering system is stable. Steady outputs' amplitudes are more than input signals' amplitudes, the input signals are amplified and steady outputs' amplitudes will decrease with vehicle velocity's increase of from figure 3's (b) and (c).

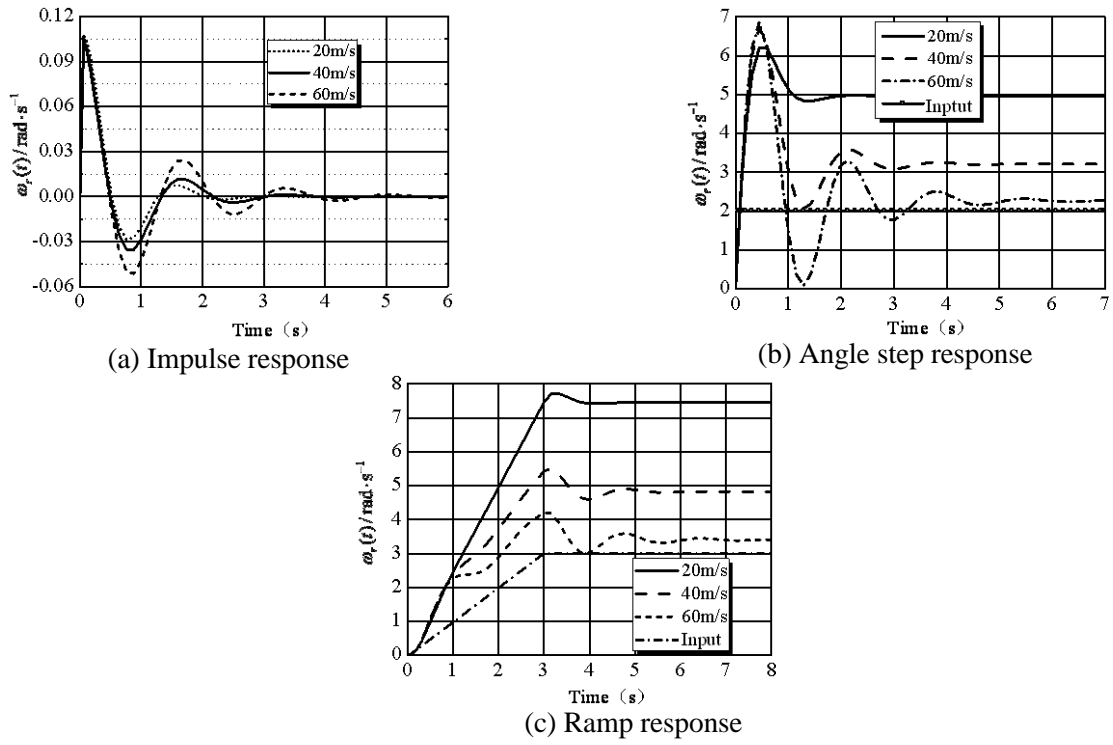


Figure 3. Steering system's steady response

From expression (15) and (17), we know that ξ and ω_n is two important parameters that affect dynamic characteristic of yaw angle velocity's instantaneous response. Yaw angle 4's (a) and (b).

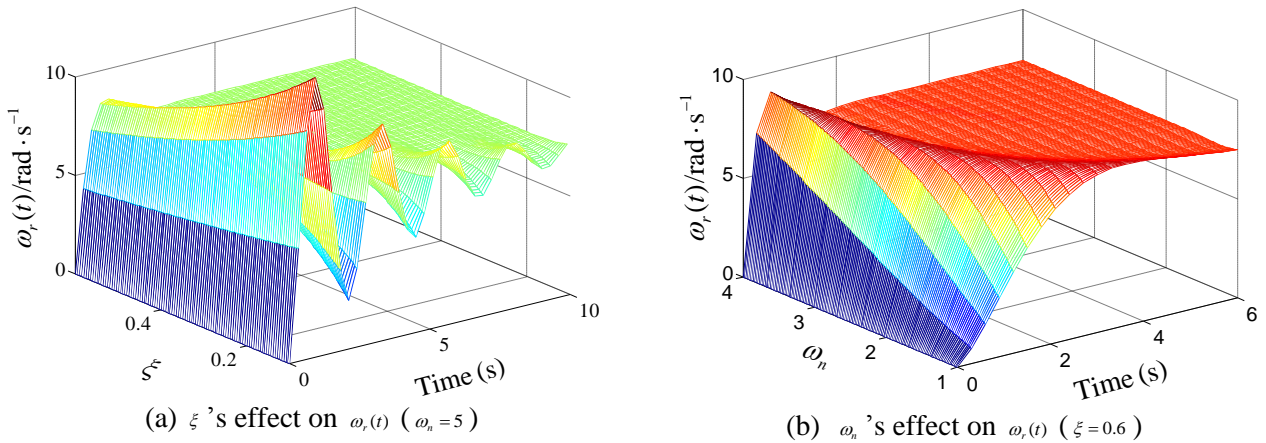


Figure 4. Effect of ξ and ω_n on $\omega_r(t)$

From figure 4, we know that dynamic response's overshoot and adjust time will be decreasing with ξ 's increasing when ω_n is determined, but ξ hardly affects rise time and peak time. From figure 6, we know that dynamic response's rise time and peak time will be decreasing with ω_n 's increasing when ξ is determined, but ω_n hardly affects overshoot. So, ξ mainly affects dynamic response's overshoot and ω_n mainly affects dynamic response's rise time and peak time.

6. Conclusion

Analyzing two-degree-of-freedom vehicle's steering motion, modeling steering dynamic system, studying steering system's stability and three steering characteristics, and studying yaw angle velocity's instantaneous response, the following conclusions are drawn through simulation:

(1) The steering system is stable. Steady outputs' amplitudes are more than input signals' amplitudes, the input signals are amplified and steady outputs' amplitudes will decrease with vehicle velocity's increase. The yaw angle velocity gain is proportional to vehicle's velocity.

(2) The over steering vehicle's yaw angle velocity gain curve is lower than the neutral steering vehicle's gain curve. The over steering vehicle's gain curve shape is up convex. The under steering vehicle's yaw angle velocity gain curve is higher than the neutral steering vehicle's gain curve. The under steering vehicle's gain curve shape is down concave.

(3) Damping ratio mainly affects dynamic response's overshoot. Yaw angle velocity's dynamic response's overshoot and adjust time will be decreasing with damping ratio's increasing when natural frequency is determined. Natural frequency mainly affects dynamic response's rise time and peak time. Yaw angle velocity's dynamic response's rise time and peak time will be decreasing with natural frequency's increasing when damping ratio is determined.

7. References

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