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Positive Solutions for P-Laplacian Discrete Boundary Value Problems via Three Critical Points Theorem

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Abstract. In this paper**,** the existence of multiple positive solutions for a class of p-Laplacian discrete boundary value problems is studied by applying three critical points theorem

Keywords: discrete boundary value problem, p-laplacian, positive solutions, three critical point theore

1. Introduction

Consider the following discrete boundary value problem

$$\begin{cases} \Delta[\phi_p(\Delta u(t-1))] + \lambda f(t,u(t)) = 0, t \in Z(1,T), \\ u(0) = 0, u(T+1) = 0, \end{cases} \tag{1}$$

Where T is a positive integer, $p > 1$ is a constant, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\phi_p(s)$ is a p-Laplacican operator, that is, $\phi_p(s) = |s|^{p-2} s$. For any $t \in Z(1,T)$, $f(t,x)$ is a continuous function on x. A sequence $\{u(t)\}_{t=0}^{T+1}$ is called a positive solution of (1) if $\{u(t)\}_{t=0}^{T+1}$ satisfies (1) and $u(t) > 0$ for $t \in Z(1,T)$.

Due to its applications in physics, such as non-Newtonian fluid mechanics, turbulence of porous media, positive solutions of p-Laplacian discrete boundary value problems are studied by many authors. Usually, these results are obtained by applying fixed point theorem and critical point theory. One can see [1-7]. Very recently, three critical points theorem has been applied to studymultiple solutions of p-Laplacian discrete boundary value problems [8,9]. Inspired by these results, in this paper we will apply a version of three critical point theorem to study the multiple positive solutions of (1).

Let E be the set of the functions $u : Z(0,T+1) \to R$ satisfying $u(0) = 0, u(T+1) = 0$. Equipped with inner product $(u,v) = \sum_{t=1}^{T} u(t)v(t), \forall u,v \in E$ and induced norm

$$\|u\| = \left(\sum_{t=1}^{T} u^2(t) \right)^{1/2}, \forall u \in E,$$

E is a T-dimensional Hilbert space..

Furthermore, for any constant $p > 1$, we define another norm

$$\|u\|_p = \left(\sum_{t=1}^{T+1} |\Delta u(t-1)|^p \right)^{1/p}, \forall u \in E.$$

Since E is finite dimensional, there are two constants C_1, $C_2 > 0$ such that

$$C_1 \|u\| \le \|u\|_p \le C_2 \|u\|. \tag{2}$$

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For convenience, we define the following two functionals

$$\Phi(u)=\frac{1}{p}\sum_{t=1}^{T+1}\left|\Delta u(t-1)\right|^{p},\ J(u)=\sum_{t=1}^{T}F(t,u(t)),\ ,$$

where $u\in E$, $F(t,x)=\int_0^x f(t,s)ds$ for any $x\in R$. Clearly, $\Phi, J\in C^1(E,R)$, that is, Φ, J are continuously differentiable on E . Using the summation by parts formula and the fact that $u(0)=0, u(T+1)=0$, it is easy to see that for any $u,v\in E$,

$$\begin{aligned}\Phi'(u)(v) &= \lim_{t\to 0}\frac{\Phi(u+tv)-\Phi(u)}{t}=\sum_{t=1}^{T+1}\left|\Delta u(t-1)\right|^{p-2}\Delta u(t-1)\Delta v(t-1)\\ &=\sum_{t=1}^{T+1}\phi_p(\Delta u(t-1))\Delta v(t-1)=\sum_{t=1}^{T}\phi_p(\Delta u(t-1))\Delta v(t-1)-\phi_p(\Delta u(T))\Delta v(T)\\ &=\phi_p(\Delta u(t-1))\Delta v(t-1)\,|_1^{T+1}-\sum_{t=1}^{T}\Delta\phi_p(\Delta u(t-1))v(t)-\phi_p(\Delta u(T))\Delta v(T)\\ &=-\sum_{t=1}^{T}\Delta\phi_p(\Delta u(t-1))v(t)\ .\end{aligned}$$

Noticing the fact that $u(0)=0, u(T+1)=0$, for any $u\in E$, we obtain

$$J'(u)(v)=\lim_{t\to 0}\frac{J(u+tv)-J(u)}{t}=\sum_{t=1}^{T}f(t,u(t))v(t)$$

for any $u,v\in E$. If

$$(\Phi-\lambda J)'(u)(v)=-\sum_{t=1}^{T}[\Delta[\phi_p(\Delta u(t-1))]+\lambda f(t,u(t))]v(t)=0,$$

then for any $t\in Z(1,T)$ and $u(0)=0, u(T+1)=0$ we have

$$\Delta[\phi_p(\Delta u(t-1))]+\lambda f(t,u(t))=0,$$

that is, a critical point of functional $\Phi-\lambda J$ corresponds to a solution of (1). Therefore, we reduce the existence of a solution of (1) to the existence of a critical point of functional $\Phi-\lambda J$ on E .

The following theorem and lemmas play an important role in proving the main result.

Suppose that $Y\subset X$. *Let* $\overline{Y}^w$ be the weak closure of Y , that is, for any $F\in Y^*$, if there exists sequence $\{u_n\}\subset Y$ such that $F(u_n)\to F(u)$, then $u\in\overline{Y}^w$.

Theorem 1 ([10]) Let X be a reflexive separable real Banach space. $\Phi: X\to R$ is a nonnegative continuous Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous invers on X^* . $J: X\to R$ is a continuous Gateaux differentiable functional whose Gateaux derivative is compact. Assume that there exists $u_0\in X$ such that $\Phi(u_0)=J(u_0)=0$ and that $\lambda\in[0,+\infty)$,

(i) $\lim_{\|u\|\to+\infty}(\Phi(u)-\lambda J(u))=+\infty;$

Furthermore, assume that there are $u_1\in X$ and $r>0$ such that

(ii) $r<\Phi(u_1)$;

(iii) $\sup_{u\in\overline{\Phi^{-1}((-\infty,r])}^w}J(u)<\frac{r}{r+\Phi(u_1)}J(u_1)$.

Then, for each $\lambda\in\Lambda_1$,where

$$\Lambda_1=\left(\frac{\Phi(u_1)}{J(u_1)-\sup_{u\in\overline{\Phi^{-1}((-\infty,r])}^w}J(u)},\frac{r}{\sup_{u\in\overline{\Phi^{-1}((-\infty,r])}^w}J(u)}\right),$$

functional $\Phi-\lambda J$ has at least three solutions in X , and moreover, for each $h>1$, there exists an open interval

$$\Lambda_2 \subset \left[0, \frac{hr}{r\frac{J(u_1)}{\Phi(u_1)} - \sup_{u \in \overline{\Phi^{-1}((-\infty, r])}^{w}} J(u)} \right]$$

and a positive constant σ such that for each $\lambda \in \Lambda_2$, functional $\Phi - \lambda J$ has at least three solutions in X whose norms are less than σ.

Lemma 1 For any $u \in E$ and $p > 1$, the following inequality holds:

$$\max_{t \in Z(1,T)} \{|u(t)|\} \le \frac{(T+1)^{(p-1)/p}}{2} \|u\|_p .$$

Proof. Suppose that there exists $k \in Z(1,T)$ such that

$|u(k)| = \max_{t \in Z(1,T)} \{|u(t)|\}$.

It follows easily from $u(0) = 0, u(T+1) = 0$ that

$$|u(k)| \le \sum_{t=1}^{k} |\Delta u(t-1)|, \quad |u(k)| \le \sum_{t=k+1}^{T+1} |\Delta u(t-1)|$$

holds, that is,

$$|u(k)| \le \frac{1}{2} \sum_{t=1}^{T+1} |\Delta u(t-1)| .$$

Discrete Holder inequality shows that

$$\sum_{t=1}^{T+1} |\Delta u(t-1)| \le (T+1)^{(p-1)/p} \left(\sum_{t=1}^{T+1} |\Delta u(t-1)|^p \right)^{1/p} = (T+1)^{(p-1)/p} \|u\|_p$$

The proof is complete.

Lemma 2 ([11]) If

$$\begin{cases} -\Delta[\phi_p(\Delta u(t-1))] \ge 0, t \in Z(1,T), \\ u(0) \ge 0, u(T+1) \ge 0 \end{cases}$$

holds, then either u is positive or $u \equiv 0$ on $Z(1,T)$.

2. Proof of main result

Theorem 2 Suppose that $f : Z(1,T) \times [0,+\infty) \to [0,+\infty)$, $f(t,0) = 0$. Suppose that there exist four constants c, d, μ, α with $c < \left(\frac{T+1}{2} \right)^{(p-1)/p} d$ and $1 < \alpha < p$ such that

(A) $\max_{t \in Z(1,T)} F(t,c) < \frac{(2c)^p}{T[(2c)^p + 2(T+1)^{p-1} d^p]} \sum_{t=1}^{T} F(t,d)$,

(B) $F(t,x) \le \mu(1 + |x|^{\alpha})$.

In addition, let

$$\varphi_1 = \frac{p(T+1)^{p-1} T \max_{t \in Z(1,T)} F(t,c)}{(2c)^p}, \varphi_2 = \frac{p\left[\sum_{t=1}^{T} F(t,d) - T \max_{t \in Z(1,T)} F(t,c) \right]}{2d^p},$$

and for any $h > 1$,

$$a = \frac{h(2cd)^p}{2^{p-1} pc^p \sum_{t=1}^{T} F(t,d) - T(T+1)^{p-1} pd^p \max_{t \in Z(1,T)} F(t,c)},$$

then for any $\lambda \in \Lambda_1 = \left(\frac{1}{\varphi_2}, \frac{1}{\varphi_1}\right)$, the problem (1) has at least two positive solutions in E, and for any $h > 1$, there exist an open interval $\Lambda_2 \subset [0, a]$ and a positive constant σ such that for $\lambda \in \Lambda_2$, the problem (1) has at least two positive solutions on E whose norms are less that σ.

Proof. Let X be the finite dimensional Hilbert space E. Then $X = X^*$. The definition of Φ shows that Φ is nonnegative continuous Gateaux differentiable and weak low semicontinuous functional, whose Gateaux derivative has continuous inverse on E, and J is a continuous Gateaux differentiable functional, whose Gateaux derivative is compact. Now for any $t \in Z(0, T+1)$, it is easy to know $u_0 = 0 \in X$ and $\Phi(u_0) = J(u_0) = 0$. In the rest of the proof, we replace X by E.

Because the solution of boundary value problems (1) is required to be positive, we suppose that $f(t,u) = 0$ for $u < 0$. We still use $f(t,u)$ and $F(t,u)$ to denote new $f(t,u)$ and $F(t,u)$. Next, considering (2) and condition (B), for any $u \in E$ and $\lambda \ge 0$,

$$\Phi(u) - \lambda J(u) = \frac{1}{p}\sum_{t=1}^{T+1} |\Delta u(t-1)|^p - \lambda \sum_{t=1}^{T} F(t, u(t))$$

$$\ge \frac{1}{p}\| u\|_p^p - \lambda\mu \sum_{t=1}^{T}(1 + |u(t)|^\alpha) \ge \frac{C_1^p}{p}\| u\|^p - \lambda\mu C_3^\alpha \| u\|^\alpha - \lambda\mu T,$$

where C_3 is such that $\|u\|_{1\alpha} \le C_3 \|u\|$,

$$\|u\|_{1\alpha} = \left(\sum_{t=1}^{T} |u(t)|^\alpha\right)^{1/\alpha}.$$

Because $\alpha < p$, for all $\lambda \in (0, +\infty)$,

$$\lim_{\|u\| \to +\infty} (\Phi(u) - \lambda J(u)) = +\infty,$$

the conditions (i) of theorem 1 is satisfied.

Let

$$u_1(t) = \begin{cases} 0, & t = 0 \text{ or } T+1, \\ d, & t \in Z(1,T), \end{cases} \quad r = \frac{(2c)^p}{p(T+1)^{p-1}}.$$

Clearly, for $u_1 \in E$,

$$\Phi(u_1) = \frac{1}{p}\sum_{t=1}^{T+1} |\Delta u_1(t-1)|^p = \frac{2d^p}{p}, \quad J(u_1) = \sum_{t=1}^{T} F(t, u_1(t)) = \sum_{t=1}^{T} F(t, d).$$

Notice that $c < \left(\frac{T+1}{2}\right)^{(p-1)/p} d$, we have

$$\Phi(u_1) = \frac{2d^p}{p} > \frac{(2c)^p}{p(T+1)^{p-1}} = r,$$

the conditions (ii) of theorem 1 is satisfied. Next we prove that condition (iii) of theorem 1 is satisfied. For any $t \in Z(1,T)$, the estimation $\Phi(u) \le r$ shows that

$$|u(t)|^p \le \frac{(T+1)^{p-1}}{2^p}\|u\|_p^p = \frac{p(T+1)^{p-1}}{2^p}\Phi(u) \le \frac{pr(T+1)^{p-1}}{2^p},$$

It follow from the definition of r that

$$\Phi^{-1}((-\infty, r]) \subseteq \{u \in E : |u(t)| \le c, \forall t \in Z(1,T)\}.$$

So, for any $u \in E$, the following result

$$\sup_{u \in \overline{\Phi^{-1}((-\infty, r])}^{w}} J(u) = \sup_{u \in \overline{\Phi^{-1}((-\infty, r])}} J(u) \le T \max_{t \in Z(1,T)} F(t, c)$$

exists. On the other hand, it is easy to know

$$\frac{r}{r + \Phi(u_1)} J(u_1) = \frac{(2c)^p}{(2c)^p + 2(T+1)^{p-1} d^p}\sum_{t=1}^{T} F(t, d).$$

It follows from the hypothesis (A) that

$$\sup_{u\in\overline{\Phi^{-1}((-\infty,r])}^{w}} J(u) \le \frac{r}{r+\Phi(u_1)} J(u_1) .$$

The conditions (iii) of theorem 1 is satisfied. Notice that

$$\frac{\Phi(u_1)}{J(u_1)-\sup_{u\in\overline{\Phi^{-1}((-\infty,r])}} J(u)} \le \frac{2d^p}{p\left[\sum_{t=1}^{T} F(t,d) - T\max_{t\in Z(1,T)} F(t,c)\right]} = \frac{1}{\varphi_2},$$

$$\frac{r}{\sup_{u\in\overline{\Phi^{-1}((-\infty,r])}} J(u)} \ge \frac{(2c)^p}{p(T+1)^{p-1} T \max_{t\in Z(1,T)} F(t,c)} = \frac{1}{\varphi_1}.$$

Simple calculation shows that $\varphi_2 > \varphi_1$.For any $\lambda \in \Lambda_1 = \left(\frac{1}{\varphi_2},\frac{1}{\varphi_1}\right)$, applying theorem 1, problem (1) has at least three solutions in E. By lemma 2, problem (1) has at least two positive solutions.

For every $h > 1$it is easy to know that

$$\frac{hr}{r\frac{J(u_1)}{\Phi(u_1)} - \sup_{u\in\overline{\Phi^{-1}((-\infty,r])}} J(u)} \le \frac{h(2cd)^p}{2^{p-1}pc^p \sum_{t=1}^{T} F(t,d) - T(T+1)^{p-1}pd^p \max_{t\in Z(1,T)} F(t,c)} = a$$

Considering condition (A), we know that $a > 0$. Then from theorem 1, for every $h > 1$, there exist an open interval $\Lambda_2 \subset [0,a]$ and a positive constant σ such that for $\lambda \in \Lambda_2$, the problem (1) has at least three solutions in E whose norms are less than σ. By lemma 2, problem (1) has at least two positive solutions whose norms are less than σ.

Next An example is given to demonstrate the result of theorem 2.

Example. Consider the case that $f(t,u) = tg(u)$, $T = 20, p = 3$, where

$$g(u) = \begin{cases} e^u - 1, & u \le 6d, \\ u + e^{6d} - 6d - 1, & u > 6d, \end{cases}$$

Then

$$G(u) = \begin{cases} e^u - u - 1, & u \le 6d, \\ \frac{1}{2}u^2 + (e^{6d} - 6d - 1)u - 6d(e^{6d} - 1) + 18d^2, & u > 6d. \end{cases}$$

It is easy to know that for $c = 1$, $d = 20$, $\mu = 20 \times e^{120}$, $\alpha = 2$, the conditions of theorem 2 are satisfied. Therefore problem (1) has at least two positive solutions.

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4. References

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