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# A New Generalized Gronwall-Bellman Type Inequality

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**Abstract.** In this paper, a new nonlinear integral inequality is established, which provide a handy tool for analyzing the global existence of solutions of differential and integral equations.

**Keywords:** Integral inequality; Global existence; Integral equation; Differential equation

## 1. Introduction

During the past decades, with the development of the theory of differential and integral equations, a lot of integral inequalities, for example [1-13], have been discovered, which play an important role in the research of boundedness, global existence, stability of solutions of differential and integral equations.

In [12], Jiang proved the following theorem:

*Theorem A:* $R_+ = [0,\infty)$. Suppose that $x(t), f(t), h(t) \in C(R_+, R_+)$. Then the following form of delay integral inequality:

$$x^p(t) \le C + \int_0^t [f(s)x^p(s) + h(s)x^p(\sigma(s))]ds, t \in R_+$$

with the initial condition

$$x(t) = \phi(t), t \in [\alpha, 0], \phi(\sigma(t)) \le (\rho(t))^{\frac{1}{p}}, t \in R_+, \sigma(t) \le 0$$

where $p, q$ are constants, $p > 0, q > 0, p \ne q$ . $\sigma(t) \in C(R_+, R)$ , $\sigma(t) \le t, -\infty < \alpha = \inf\{\sigma(t), t \in R_+\} \le 0, \phi \in C([\alpha,0], R_+)$ , implies that

$$x(t) \le \exp(\int_0^t \frac{f(s)}{p} ds)\ [C^{\frac{p-q}{p}} + \int_0^t \frac{p-q}{p} h(s) \exp(\int_0^s \frac{p-q}{p} f(\tau)d\tau)ds]^{\frac{1}{p-q}}$$

for $t \in [0, t_0]$, where $t_0$ is a positive number satisfying

$$\inf_{t\in[0,t_0]} \{C^{\frac{p-q}{p}} + \int_0^t \frac{p-q}{p} h(s)\exp(\int_0^s \frac{p-q}{p} f(\tau)d\tau)ds\} > 0 \cdot$$

In this paper, motivated by the above work, we will prove more general theorem and establish a new integral inequality. Also we will give one example so as to illustrate the validity of the present integral inequality.

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## 2. Main Results

*Theorem 2.1:* Assume that $x, a \in C(R_+, R_+)$ , $a(t)$ is non-decreasing. $f, g, \partial_t f, \partial_t g \in C(R_+ \times R_+, R_+)$ . $\omega \in C(R_+, R_+)$ be nondecreasing with $\omega(u) > 0$ on $(0, \infty)$ . If $x(t)$ satisfies the following delay integral inequality:

$$x^p(t) \le a(t) + \int_0^t [f(s,t)x^p(\sigma_1(s)) + g(s,t)x^q(\sigma_2(s))\omega(x(\sigma_3(s)))]ds,\ t \in R_+ \tag{1}$$

with the initial condition

$$x(t) = \phi(t), t \in [\alpha, 0], \phi(\sigma_i(t)) \le (a(t))^{\frac{1}{p}}, t \in R_+, \sigma_i(t) \le 0, i = 1, 2, 3 \tag{2}$$

where $p, q$ are constants, $p > q > 0$ , $\sigma_i \in C(R_+, R)$ , $\sigma_i(t) \le t, -\infty < \alpha = \inf\{\min\{\sigma_i(t), i = 1,2,3\}, t \in R_+\} \le 0, \phi \in C([\alpha, 0], R_+)$ , then for $t \in R_+$,

$$x(t) \le \{\Omega^{-1}[\Omega(\exp(\frac{p-q}{p}F_1(t))a^{\frac{p-q}{p}}(t)) + \exp(\frac{p-q}{p}F_1(t))\int_0^t \frac{p-q}{p}F_2{}'(s)\exp(-\frac{p-q}{p}F_1(s))ds]\}^{\frac{1}{p-q}} \tag{3}$$

where

$$F_1(t) = \int_0^t f(s,t)ds, F_2(t) = \int_0^t g(s,t)ds \ , \tag{4}$$

$\Omega(r) = \int_1^r \frac{1}{\omega(s^{\frac{1}{p-q}})}ds, \Omega^{-1}$ is the inverse of $\Omega$.

*Proof:* We notice (3) holds for t = 0 obviously. Let the right side of (1) be $\varphi^p(t)$. Then

$$x(t) \le \varphi(t) \tag{5}$$

When $\sigma_i(t) \ge 0$, we have

$$x(\sigma_i(t)) \le \varphi(\sigma_i(t)) \le \varphi(t) \tag{6}$$

When $\sigma_i(t) \le 0$, we have

$$x(\sigma_i(t)) = \phi(\sigma_i(t)) \le a^{\frac{1}{p}}(t) \le \varphi(t) \tag{7}$$

So from (6), (7) we always have $x(\sigma_i(t)) \le \varphi(t), i = 1, 2, 3$ . Fix $T > 0$. Then for $t \in (0, T]$, we have

$$\varphi^p(t) \le a(T) + \int_0^t [f(s,t)\varphi^p(s) + g(s,t)\varphi^q(s)\omega(\varphi(s))]ds \tag{8}$$

Let the right side of (8) be $u^p(t)$. Then

$$\varphi(t) \le u(t), t \in (0, T] \tag{9}$$

and

$$u^{p-1}(t)u'(t)=\frac{1}{p}[\int_0^t\frac{\partial f(s,t)}{\partial t}\varphi^p(s)ds+f(t,t)\varphi^p(t)+\int_0^t\frac{\partial g(s,t)}{\partial t}\varphi^q(s)\omega(\varphi(s))ds+g(t,t)\varphi^q(t)\omega(\varphi(t))]$$

$$\le\frac{d\int_0^t f(s,t)ds}{dt}\frac{1}{p}\varphi^p(t)+\frac{d\int_0^t g(s,t)\omega(u(s))ds}{dt}\frac{1}{p}\varphi^q(t)$$

$$\le\frac{d\int_0^t f(s,t)ds}{dt}\frac{1}{p}u^p(t)+\frac{d\int_0^t g(s,t)\omega(u(s))ds}{dt}\frac{1}{p}u^q(t) \tag{10}$$

Then

$$u'(t)\le\frac{d\int_0^t f(s,t)ds}{dt}\frac{1}{p}u(t)+\frac{d\int_0^t g(s,t)\omega(u(s))ds}{dt}\frac{1}{p}u^{1+q-p}(t) \tag{11}$$

Let $v(t)=u^{p-q}(t)$. Then

$$v'(t)\le\frac{d\int_0^t f(s,t)ds}{dt}\frac{p-q}{p}v(t)+\frac{d\int_0^t g(s,t)\omega(u(s))ds}{dt}\frac{p-q}{p}\le\frac{d\int_0^t f(s,t)ds}{dt}\frac{p-q}{p}v(t)+\frac{d\int_0^t g(s,t)ds}{dt}\frac{p-q}{p}\omega(v^{\frac{1}{p-q}}(t))$$

that is,

$$v'(t)-\frac{p-q}{p}F_1'(t)v(t)\le\frac{p-q}{p}F_2'(t)\omega(v^{\frac{1}{p-q}}(t)) \tag{12}$$

Multiplying $\exp(-\frac{p-q}{p}F_1(t))$ on both sides of (12), we have

$$\frac{d[\exp(-\frac{p-q}{p}F_1(t))v(t)]}{dt}\le\frac{p-q}{p}F_2'(t)\omega(v^{\frac{1}{p-q}}(t))\exp(-\frac{p-q}{p}F_1(t)) \tag{13}$$

Integrating (13) from $0$ to $t$, it follows

$$\exp(-\frac{p-q}{p}F_1(t))v(t)-v(0)\le\int_0^t\frac{p-q}{p}F_2'(s)\omega(v^{\frac{1}{p-q}}(s))\exp(-\frac{p-q}{p}F_1(s))ds \tag{14}$$

Since $v(0)=u^{p-q}(0)=a^{\frac{p-q}{p}}(T)$, we have

$$v(t)\le\{a^{\frac{p-q}{p}}(T)+\int_0^t\frac{p-q}{p}F_2'(s)\omega(v^{\frac{1}{p-q}}(s))\exp(-\frac{p-q}{p}F_1(s))ds\}\exp(\frac{p-q}{p}F_1(t)) \tag{15}$$

and

$$u(t)\le\exp(\frac{1}{p}F_1(t))\ \{a^{\frac{p-q}{p}}(T)+\int_0^t\frac{p-q}{p}F_2'(s)\omega(u(s))\exp(-\frac{p-q}{p}F_1(s))ds\}^{\frac{1}{p-q}} \tag{16}$$

Let $k(t)=a^{\frac{p-q}{p}}(T)+\int\limits_0^t \frac{p-q}{p}F_2'(s)\omega(v^{\frac{1}{p-q}}(s))\exp(-\frac{p-q}{p}F_1(s))ds$.

Then

$$u(t)\le \exp(\frac{1}{p}F_1(t))k^{\frac{1}{p-q}}(t) \tag{17}$$

and

$$k'(t)=\frac{p-q}{p}F_2'(t)\omega(u(t))\exp(-\frac{p-q}{p}F_1(t))\le \frac{p-q}{p}F_2'(t)\omega(\exp(\frac{1}{p}F_1(T))k^{\frac{1}{p-q}}(t))\exp(-\frac{p-q}{p}F_1(t))$$

Then

$$\frac{k'(t)}{\omega(\exp(\frac{1}{p}F_1(T))k^{\frac{1}{p-q}}(t))}\le \frac{p-q}{p}F_2'(t)\exp(-\frac{p-q}{p}F_1(t)) \tag{18}$$

Integrating (18) from $0$ to $t$, it follows

$$\Omega[\exp(\frac{p-q}{p}F_1(T))k(t)]-\Omega[\exp(\frac{p-q}{p}F_1(T))a^{\frac{p-q}{p}}(T)]\le \exp(\frac{p-q}{p}F_1(T))\int\limits_0^t \frac{p-q}{p}F_2'(s)\exp(-\frac{p-q}{p}F_1(s))ds \tag{19}$$

Then

$$k(t)\le \exp(\frac{q-p}{p}F_1(T))\Omega^{-1}[\Omega(\exp(\frac{p-q}{p}F_1(T))a^{\frac{p-q}{p}}(T))+\exp(\frac{p-q}{p}F_1(T))\int\limits_0^t \frac{p-q}{p}F_2'(s)\exp(-\frac{p-q}{p}F_1(s))ds] \tag{20}$$

From (17) and (20), it follows

$$u(t)\le \{\Omega^{-1}[\Omega(\exp(\frac{p-q}{p}F_1(T))a^{\frac{p-q}{p}}(T))+\exp(\frac{p-q}{p}F_1(T))\int\limits_0^t \frac{p-q}{p}F_2'(s)\exp(-\frac{p-q}{p}F_1(s))ds]\}^{\frac{1}{p-q}}, t\in(0,T] \tag{21}$$

Combining (5), (9), (21) we have

$$x(t)\le \{\Omega^{-1}[\Omega(\exp(\frac{p-q}{p}F_1(T))a^{\frac{p-q}{p}}(T))+\exp(\frac{p-q}{p}F_1(T))\int\limits_0^t \frac{p-q}{p}F_2'(s)\exp(-\frac{p-q}{p}F_1(s))ds]\}^{\frac{1}{p-q}}, t\in(0,T] \tag{22}$$

Setting $t=T$ and considering $T\in R_+$ is arbitrary, we have completed the proof.

Corallary 2.2: Assume that $x,a,f,g\in C(R_+,R_+), m\in C^1(R_+,R_+)$. If $x,a,p,q,\sigma_i(t),\alpha,\phi,\omega$ are the same as in Theorem 2.1, and $x(t)$ satisfies the following delay integral inequality:

$$x^p(t)\le a(t)+m(t)\int\limits_0^t [f(s)x^p(\sigma_1(s))+g(s)x^q(\sigma_2(s))\omega(x(\sigma_3(s)))]ds,\ t\in R_+ \tag{23}$$

with the initial condition (2), then for $t\in R_+$,

$$x(t)\le \{\Omega^{-1}[\Omega(\exp(\frac{p-q}{p}F_1(t))a^{\frac{p-q}{p}}(t))+\exp(\frac{p-q}{p}F_1(t))\int\limits_0^t \frac{p-q}{p}F_2'(s)\exp(-\frac{p-q}{p}F_1(s))ds]\}^{\frac{1}{p-q}} \tag{24}$$

where

$$F_1(t)=m(t)\int_0^t f(s)ds, F_2(t)=m(t)\int_0^t g(s)ds \tag{25}$$

## 3. Application

*Example:* We consider the following delay differential equation

$$(x^p(t))'=F(t,x(\sigma_1(t)),x(\sigma_2(t)),x(\sigma_3(t))) \tag{26}$$

with the initial condition

$$x(t)=\phi(t), t\in[\alpha,0], |\phi(\sigma_i(t))|\le A^{\frac{1}{p}}, t\in R_+, \sigma_i(t)\le 0, i=1,2,3$$

where $A>0$ is a constants and $A=x^p(0)$ , $F\in C(R_+\times R^3,R)$ ,. $\sigma_i(t)\in C(R_+,R)$ , $\sigma_i(t)\le t, -\infty<\alpha$ $=\inf\{\min\{\sigma_i(t), i=1,2,3\}, t\in R_+\}\le 0,\ \phi\in C([\alpha,0],R_+)$, Assume $|F(t,x,y,z)|\le f(t)|x|^p+g(t)|x|^q v(|z|)$ where $f,g,v\in C(R_+,R_+)$, $v$ is nondecreasing, and $\int_1^\infty v(s)ds=\infty\cdot\ p>q>0$

Integrating (26) from $0$ to $t$, it follows

$$x^p(t)-x^p(0)=\int_o^t F(s,x(\sigma_1(s)),x(\sigma_2(s)),x(\sigma_3(s)))ds$$

So

$$|x^p(t)|-|x^p(0)|\le|x^p(t)-x^p(0)|\le\int_o^t|F(s,x(\sigma_1(s)),x(\sigma_2(s)),x(\sigma_3(s)))|ds$$

$$\le\int_o^t[f(s)|x(\sigma_1(s))|^p+g(s)|x(\sigma_2(s))|^q\ v(|x(\sigma_3(s))|)]ds$$

Taking $\omega(u)=v(u)$, from Theorem 2.1, we can reach the estimate

$$|x(t)|\le\{\Omega^{-1}[\Omega(\exp(\frac{p-q}{p}F_1(t))|A|^{\frac{p-q}{p}})\ +\exp(\frac{p-q}{p}F_1(t))\int_0^t\frac{p-q}{p}F_2{}'(s)\exp(-\frac{p-q}{p}F_1(s))ds]\}^{\frac{1}{p-q}}$$

where

$$F_1(t)=\int_0^t f(s)ds, F_2(t)=\int_0^t g(s)ds,$$

which shows $x(t)$ does not blow up in finite time. So the solution of (26) is global.

## 4. Conclusions

In this paper, we establish a new integral inequality, which provides a handy tool in the qualitative analysis of solutions of integral equations and differential equations. Our result generalizes the result in [12].

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