

# A New Generalized Gronwall-Bellman Type Inequality

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**Abstract.** In this paper, a new nonlinear integral inequality is established, which provide a handy tool for analyzing the global existence of solutions of differential and integral equations.

**Keywords:** Integral inequality; Global existence; Integral equation; Differential equation

## 1. Introduction

During the past decades, with the development of the theory of differential and integral equations, a lot of integral inequalities, for example [1-13], have been discovered, which play an important role in the research of boundedness, global existence, stability of solutions of differential and integral equations.

In [12], Jiang proved the following theorem:

*Theorem A:*  $R_+ = [0, \infty)$ . Suppose that  $x(t), f(t), h(t) \in C(R_+, R_+)$ . Then the following form of delay integral inequality:

$$x^p(t) \leq C + \int_0^t [f(s)x^p(s) + h(s)x^p(\sigma(s))] ds, t \in R_+$$

with the initial condition

$$x(t) = \phi(t), t \in [\alpha, 0], \phi(\sigma(t)) \leq (\rho(t))^{\frac{1}{p}}, t \in R_+, \sigma(t) \leq 0$$

where  $p, q$  are constants,  $p > 0, q > 0, p \neq q$ .  $\sigma(t) \in C(R_+, R)$ ,  $\sigma(t) \leq t, -\infty < \alpha = \inf\{\sigma(t), t \in R_+\} \leq 0, \phi \in C([\alpha, 0], R_+)$ , implies that

$$x(t) \leq \exp\left(\int_0^t \frac{f(s)}{p} ds\right) \left[ C^{\frac{p-q}{p}} + \int_0^t \frac{p-q}{p} h(s) \exp\left(\int_0^s \frac{p-q}{p} f(\tau) d\tau\right) ds \right]^{\frac{1}{p-q}}$$

for  $t \in [0, t_0]$ , where  $t_0$  is a positive number satisfying

$$\inf_{t \in [0, t_0]} \left\{ C^{\frac{p-q}{p}} + \int_0^t \frac{p-q}{p} h(s) \exp\left(\int_0^s \frac{p-q}{p} f(\tau) d\tau\right) ds \right\} > 0.$$

In this paper, motivated by the above work, we will prove more general theorem and establish a new integral inequality. Also we will give one example so as to illustrate the validity of the present integral inequality.

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## 2. Main Results

*Theorem 2.1:* Assume that  $x, a \in C(R_+, R_+)$ ,  $a(t)$  is non-decreasing.  $f, g, \partial_t f, \partial_t g \in C(R_+ \times R_+, R_+)$ .  $\omega \in C(R_+, R_+)$  be nondecreasing with  $\omega(u) > 0$  on  $(0, \infty)$ . If  $x(t)$  satisfies the following delay integral inequality:

$$x^p(t) \leq a(t) + \int_0^t [f(s, t)x^p(\sigma_1(s)) + g(s, t)x^q(\sigma_2(s))\omega(x(\sigma_3(s)))]ds, t \in R_+ \quad (1)$$

with the initial condition

$$x(t) = \phi(t), t \in [\alpha, 0], \phi(\sigma_i(t)) \leq (a(t))^{\frac{1}{p}}, t \in R_+, \sigma_i(t) \leq 0, i = 1, 2, 3 \quad (2)$$

where  $p, q$  are constants,  $p > q > 0$ ,  $\sigma_i \in C(R_+, R)$ ,  $\sigma_i(t) \leq t, -\infty < \alpha = \inf\{\min\{\sigma_i(t), i = 1, 2, 3\}, t \in R_+\} \leq 0, \phi \in C([\alpha, 0], R_+)$ , then for  $t \in R_+$ ,

$$x(t) \leq \{\Omega^{-1}[\Omega(\exp(\frac{p-q}{p}F_1(t))a^{\frac{p-q}{p}}(t)) + \exp(\frac{p-q}{p}F_1(t))\int_0^t \frac{p-q}{p}F_2'(s)\exp(-\frac{p-q}{p}F_1(s))ds]\}^{\frac{1}{p-q}} \quad (3)$$

where

$$F_1(t) = \int_0^t f(s, t)ds, F_2(t) = \int_0^t g(s, t)ds, \quad (4)$$

$$\Omega(r) = \int_1^r \frac{1}{\omega(s^{\frac{1}{p-q}})}ds, \Omega^{-1} \text{ is the inverse of } \Omega.$$

*Proof:* We notice (3) holds for  $t = 0$  obviously. Let the right side of (1) be  $\varphi^p(t)$ . Then

$$x(t) \leq \varphi(t) \quad (5)$$

When  $\sigma_i(t) \geq 0$ , we have

$$x(\sigma_i(t)) \leq \varphi(\sigma_i(t)) \leq \varphi(t) \quad (6)$$

When  $\sigma_i(t) \leq 0$ , we have

$$x(\sigma_i(t)) = \phi(\sigma_i(t)) \leq a^{\frac{1}{p}}(t) \leq \varphi(t) \quad (7)$$

So from (6), (7) we always have  $x(\sigma_i(t)) \leq \varphi(t), i = 1, 2, 3$ . Fix  $T > 0$ . Then for  $t \in (0, T]$ , we have

$$\varphi^p(t) \leq a(T) + \int_0^t [f(s, t)\varphi^p(s) + g(s, t)\varphi^q(s)\omega(\varphi(s))]ds \quad (8)$$

Let the right side of (8) be  $u^p(t)$ . Then

$$\varphi(t) \leq u(t), t \in (0, T] \quad (9)$$

and

$$\begin{aligned}
u^{p-1}(t)u'(t) &= \frac{1}{p} \left[ \int_0^t \frac{\partial f(s,t)}{\partial t} \varphi^p(s) ds + f(t,t) \varphi^p(t) + \int_0^t \frac{\partial g(s,t)}{\partial t} \varphi^q(s) \omega(\varphi(s)) ds + g(t,t) \varphi^q(t) \omega(\varphi(t)) \right] \\
&\leq \frac{d \int_0^t f(s,t) ds}{dt} \frac{1}{p} \varphi^p(t) + \frac{d \int_0^t g(s,t) \omega(u(s)) ds}{dt} \frac{1}{p} \varphi^q(t) \\
&\leq \frac{d \int_0^t f(s,t) ds}{dt} \frac{1}{p} u^p(t) + \frac{d \int_0^t g(s,t) \omega(u(s)) ds}{dt} \frac{1}{p} u^q(t)
\end{aligned} \tag{10}$$

Then

$$u'(t) \leq \frac{d \int_0^t f(s,t) ds}{dt} \frac{1}{p} u(t) + \frac{d \int_0^t g(s,t) \omega(u(s)) ds}{dt} \frac{1}{p} u^{1+q-p}(t) \tag{11}$$

Let  $v(t) = u^{p-q}(t)$ . Then

$$v'(t) \leq \frac{d \int_0^t f(s,t) ds}{dt} \frac{p-q}{p} v(t) + \frac{d \int_0^t g(s,t) \omega(u(s)) ds}{dt} \frac{p-q}{p} \leq \frac{d \int_0^t f(s,t) ds}{dt} \frac{p-q}{p} v(t) + \frac{d \int_0^t g(s,t) ds}{dt} \frac{p-q}{p} \omega(v^{\frac{1}{p-q}}(t))$$

that is,

$$v'(t) - \frac{p-q}{p} F_1'(t) v(t) \leq \frac{p-q}{p} F_2'(t) \omega(v^{\frac{1}{p-q}}(t)) \tag{1}$$

2)

Multiplying  $\exp(-\frac{p-q}{p} F_1(t))$  on both sides of (12), we have

$$\frac{d[\exp(-\frac{p-q}{p} F_1(t)) v(t)]}{dt} \leq \frac{p-q}{p} F_2'(t) \omega(v^{\frac{1}{p-q}}(t)) \exp(-\frac{p-q}{p} F_1(t)) \tag{13}$$

Integrating (13) from 0 to  $t$ , it follows

$$\exp(-\frac{p-q}{p} F_1(t)) v(t) - v(0) \leq \int_0^t \frac{p-q}{p} F_2'(s) \omega(v^{\frac{1}{p-q}}(s)) \exp(-\frac{p-q}{p} F_1(s)) ds \tag{14}$$

Since  $v(0) = u^{p-q}(0) = a^{\frac{p-q}{p}}(T)$ , we have

$$v(t) \leq \{ a^{\frac{p-q}{p}}(T) + \int_0^t \frac{p-q}{p} F_2'(s) \omega(v^{\frac{1}{p-q}}(s)) \exp(-\frac{p-q}{p} F_1(s)) ds \} \exp(\frac{p-q}{p} F_1(t)) \tag{15}$$

and

$$u(t) \leq \exp(\frac{1}{p} F_1(t)) \{ a^{\frac{p-q}{p}}(T) + \int_0^t \frac{p-q}{p} F_2'(s) \omega(u(s)) \exp(-\frac{p-q}{p} F_1(s)) ds \}^{\frac{1}{p-q}} \tag{16}$$

Let  $k(t) = a^{\frac{p-q}{p}}(T) + \int_0^t \frac{p-q}{p} F_2'(s) \omega(v^{\frac{1}{p-q}}(s)) \exp(-\frac{p-q}{p} F_1(s)) ds$ .

Then

$$u(t) \leq \exp(\frac{1}{p} F_1(t)) k^{\frac{1}{p-q}}(t) \quad (17)$$

and

$$k'(t) = \frac{p-q}{p} F_2'(t) \omega(u(t)) \exp(-\frac{p-q}{p} F_1(t)) \leq \frac{p-q}{p} F_2'(t) \omega(\exp(\frac{1}{p} F_1(T)) k^{\frac{1}{p-q}}(t)) \exp(-\frac{p-q}{p} F_1(t))$$

Then

$$\frac{k'(t)}{\omega(\exp(\frac{1}{p} F_1(T)) k^{\frac{1}{p-q}}(t))} \leq \frac{p-q}{p} F_2'(t) \exp(-\frac{p-q}{p} F_1(t)) \quad (18)$$

Integrating (18) from 0 to  $t$ , it follows

$$\Omega[\exp(\frac{p-q}{p} F_1(T)) k(t)] - \Omega[\exp(\frac{p-q}{p} F_1(T)) a^{\frac{p-q}{p}}(T)] \leq \exp(\frac{p-q}{p} F_1(T)) \int_0^t \frac{p-q}{p} F_2'(s) \exp(-\frac{p-q}{p} F_1(s)) ds \quad (19)$$

Then

$$k(t) \leq \exp(\frac{p-q}{p} F_1(T)) \Omega^{-1}[\Omega(\exp(\frac{p-q}{p} F_1(T)) a^{\frac{p-q}{p}}(T)) + \exp(\frac{p-q}{p} F_1(T)) \int_0^t \frac{p-q}{p} F_2'(s) \exp(-\frac{p-q}{p} F_1(s)) ds] \quad (20)$$

From (17) and (20), it follows

$$u(t) \leq \{\Omega^{-1}[\Omega(\exp(\frac{p-q}{p} F_1(T)) a^{\frac{p-q}{p}}(T)) + \exp(\frac{p-q}{p} F_1(T)) \int_0^t \frac{p-q}{p} F_2'(s) \exp(-\frac{p-q}{p} F_1(s)) ds]\}^{\frac{1}{p-q}}, t \in (0, T] \quad (21)$$

Combining (5), (9), (21) we have

$$x(t) \leq \{\Omega^{-1}[\Omega(\exp(\frac{p-q}{p} F_1(T)) a^{\frac{p-q}{p}}(T)) + \exp(\frac{p-q}{p} F_1(T)) \int_0^t \frac{p-q}{p} F_2'(s) \exp(-\frac{p-q}{p} F_1(s)) ds]\}^{\frac{1}{p-q}}, t \in (0, T] \quad (22)$$

Setting  $t = T$  and considering  $T \in R_+$  is arbitrary, we have completed the proof.

Corollary 2.2: Assume that  $x, a, f, g \in C(R_+, R_+), m \in C^1(R_+, R_+)$ . If  $x, a, p, q, \sigma_i(t), \alpha, \phi, \omega$  are the same as in Theorem 2.1, and  $x(t)$  satisfies the following delay integral inequality:

$$x^p(t) \leq a(t) + m(t) \int_0^t [f(s) x^p(\sigma_1(s)) + g(s) x^q(\sigma_2(s)) \omega(x(\sigma_3(s)))] ds, t \in R_+ \quad (23)$$

with the initial condition (2), then for  $t \in R_+$ ,

$$x(t) \leq \{\Omega^{-1}[\Omega(\exp(\frac{p-q}{p} F_1(t)) a^{\frac{p-q}{p}}(t)) + \exp(\frac{p-q}{p} F_1(t)) \int_0^t \frac{p-q}{p} F_2'(s) \exp(-\frac{p-q}{p} F_1(s)) ds]\}^{\frac{1}{p-q}} \quad (24)$$

where

$$F_1(t) = m(t) \int_0^t f(s) ds, F_2(t) = m(t) \int_0^t g(s) ds \quad (25)$$

### 3. Application

*Example:* We consider the following delay differential equation

$$(x^p(t))' = F(t, x(\sigma_1(t)), x(\sigma_2(t)), x(\sigma_3(t))) \quad (26)$$

with the initial condition

$$x(t) = \phi(t), t \in [\alpha, 0], |\phi(\sigma_i(t))| \leq A^{\frac{1}{p}}, t \in R_+, \sigma_i(t) \leq 0, i = 1, 2, 3$$

where  $A > 0$  is a constants and  $A = x^p(0)$ ,  $F \in C(R_+ \times R^3, R)$ ,  $\sigma_i(t) \in C(R_+, R)$ ,  $\sigma_i(t) \leq t, -\infty < \alpha = \inf\{\min\{\sigma_i(t), i=1,2,3\}, t \in R_+\} \leq 0$ ,  $\phi \in C([\alpha, 0], R_+)$ , Assume  $|F(t, x, y, z)| \leq f(t)|x|^p + g(t)|x|^q v(|z|)$

where  $f, g, v \in C(R_+, R_+)$ ,  $v$  is nondecreasing, and  $\int_1^\infty v(s) ds = \infty \cdot p > q > 0$

Integrating (26) from 0 to  $t$ , it follows

$$x^p(t) - x^p(0) = \int_0^t F(s, x(\sigma_1(s)), x(\sigma_2(s)), x(\sigma_3(s))) ds$$

So

$$\begin{aligned} |x^p(t) - x^p(0)| &\leq \int_0^t |F(s, x(\sigma_1(s)), x(\sigma_2(s)), x(\sigma_3(s)))| ds \\ &\leq \int_0^t [f(s)|x(\sigma_1(s))|^p + g(s)|x(\sigma_2(s))|^q v(|x(\sigma_3(s))|)] ds \end{aligned}$$

Taking  $\omega(u) = v(u)$ , from Theorem 2.1, we can reach the estimate

$$|x(t)| \leq \left\{ \Omega^{-1} \left[ \Omega \left( \exp\left(\frac{p-q}{p} F_1(t)\right) |A|^{\frac{p-q}{p}} + \exp\left(\frac{p-q}{p} F_1(t)\right) \int_0^t \frac{p-q}{p} F_2'(s) \exp\left(-\frac{p-q}{p} F_1(s)\right) ds \right] \right\}^{\frac{1}{p-q}}$$

where

$$F_1(t) = \int_0^t f(s) ds, F_2(t) = \int_0^t g(s) ds,$$

which shows  $x(t)$  does not blow up in finite time. So the solution of (26) is global.

### 4. Conclusions

In this paper, we establish a new integral inequality, which provides a handy tool in the qualitative analysis of solutions of integral equations and differential equations. Our result generalizes the result in [12].

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