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Analysis of a Nonlinear Integral Inequality on Time Scales

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Abstract. In this paper, a new Volterra-Fredholm type delay integral inequality on time scales is established, which can be used as a hand tool in the investigation of qualitative properties of delay dynamic equations.

Keywords: Delay integral inequality; Time scales; Dynamic equation; Bounded

1. Introduction

During the past decades, many integral inequalities have been established since then, for example [1-10], which have played an important role in the research of qualitative properties of solutions of dynamic equations.

Our aim in this paper is to establish a new Volterra-Fredholm type delay integral inequality on time scales, which provides new bound for unknown functions.

In the rest of the paper, R denotes the set of real numbers and $R_+ = [0,\infty)$. T denotes an arbitrary time scale and $T_0 = [x_0,\infty)\bigcap T, \overline{T}_0 = [y_0,\infty)\bigcap T$, where $x_0, y_0 \in T$. The set T^κ is defined to be T if T does not have a left-scattered maximum, otherwise it is T without the left-scattered maximum. On T we define the forward and backward jump operators $\sigma \in (T,T)$ and $\rho \in (T,T)$ such that $\sigma(t) = \inf\{s \in T, s > t\}$, $\rho(t) = \sup\{s \in T, s < t\}$.

Definition 1: A point $t \in T$ with $t > infT$ is said to be left-dense if $\rho(t) = t$ and right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$.

Definition 2: A function $f \in (T,R)$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1 + \mu(t) f(t) \neq 0$, where $\mu(t) = \sigma(t) - t$. C_{rd} denotes the set of rd-continuous functions, while $\overline{R}$ denotes the set of all regressive and rd-continuous functions, and $\overline{R}^+ = \{f \mid f \in \overline{R}, 1 + \mu(t) f(t) > 0, \forall t \in T\}$.

Definition 3: For some $t \in T^\kappa$, and a function $f \in (T,R)$, the delta derivative of f is denoted by $f^\Delta(t)$, and satisfies $| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) | \le \varepsilon | \sigma(t) - s |$ for $\forall \varepsilon > 0$, where $s \in U$, and U is a neighborhood of t. The function f is called delta differential on T^κ

Similarly, for some $y \in T^\kappa$, and a function $f \in (T \times T, R)$, the partial delta derivative of f with respect to y is denoted by $(f(x,y))_y^\Delta$, and satisfies

$| f(x,\sigma(y)) - f(x,s) - (f(x,y))_y^\Delta(\sigma(y) - s) | \le \varepsilon | \sigma(y) - s |$ for $\forall \varepsilon > 0$, where $s \in U$, and U is a neighborhood of y.

Definition 4: For some $a, b \in T$ and a function $f \in (T,R)$, the Cauchy integral of f is defined by

$\int_a^b f(t)\Delta t = F(b) - F(a)$, where $F^\Delta(t) = f(t), t \in T^\kappa$.

Similarly, for some $a, b \in T$ and a function $f \in (T \times T, R)$, the Cauchy partial integral of f with respect to

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y is defined by $\int_a^b f(x,y)\Delta y = F(x,b) - F(x,a)$, where $(F(x,y))_y^\Delta = f(x,y), y \in T^\kappa$.

More details on time scales can be referred to [11].

2. Main Results

We will give some lemmas for further use.

Lemma 2.1 ([11], Gronwall's inequality): Suppose $X \in T_0$ is an arbitrarily fixed number, and $u(X,y)$, $b(X,y) \in C_{rd}$, $m(X,y) \in \overline{R}^+$ with respect to y, $m(X,y) \geq 0$, then

$$u(X,y) \leq b(X,y) + \int_{y_0}^{y} m(X,t)u(X,t)\Delta t, y \in \overline{T}_0$$

implies

$$u(X,y) \leq b(X,y) + \int_{y_0}^{y} e_m(y,\sigma(t))b(X,t)m(X,t)\Delta t, y \in \overline{T}_0,$$

where $e_m(y,y_0)$ is the unique solution of the following equation

$$(z(X,y))_y^\Delta = m(X,y)z(X,y), z(X,y_0) = 1.$$

Lemma 2.2 [15]: Assume that $a \geq 0, p \geq q \geq 0$, and $p \neq 0$, then for any $K > 0$,

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}.$$

Theorem 2.1: Suppose $u, f_i, g_i, h_i \in C_{rd}(T_0 \times \overline{T}_0, R_+), i = 1,2$, p,q,r,m,C m,C are constants, and $p \geq q \geq 0, p \geq r \geq 0,$ $p \geq m \geq 0,$ $p \neq 0, C > 0$, $\tau_1 \in (T_0,T), \tau_1(x) \leq x, -\infty < \alpha = \inf\{\tau_1(x), x \in T_0\} \leq x_0$, $\tau_2 \in (\overline{T}_0,T), \tau_2(y) \leq y, -\infty < \beta = \inf\{\tau_2(y), y \in \overline{T}_0\} \leq y_0$, $\phi \in C_{rd}(([\alpha,x_0] \times [\beta,y_0]) \cap T^2, R_+)$, $M \in T_0, N \in \overline{T_0}$ are two fixed numbers, $K > 0$ is an arbitrary constant. If for $(x,y) \in ([x_0,M] \cap T \times ([y_0,N] \cap T)$, $u(x,y)$ satisfies the following inequality

$$\begin{aligned} u^p(x,y) \leq {} & C + \int_{y_0}^{y}\int_{x_0}^{x} [f_1(s,t)u^q(\tau_1(s),\tau_2(t)) + g_1(s,t)u^r(\tau_1(s),\tau_2(t))]\Delta s\Delta t + \int_{y_0}^{y}\int_{x_0}^{x}\int_{y_0}^{t}\int_{x_0}^{s} h_1(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t \\ & + \int_{y_0}^{N}\int_{x_0}^{M} [f_2(s,t)u^q(\tau_1(s),\tau_2(t)) + g_2(s,t)u^r(\tau_1(s),\tau_2(t))]\Delta s\Delta t + \int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s} h_2(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t \end{aligned} \tag{1}$$

with the initial condition

$$\begin{cases} u(x,y) = \phi(x,y), x \in [\alpha,x_0] \cap T, or, y \in [\beta,y_0] \cap T \\ \phi(\tau_1(x),\tau_2(y)) \leq C^{\frac{1}{p}}, \tau_1(x) \leq x_0, or, \tau_2(y) \leq y_0 \end{cases}, \tag{2}$$

then

$$u(x,y) \leq \{[\frac{\overline{C} + B_6}{1 - B_5}]B_3(x,y) + B_4(x,y)\}^{\frac{1}{p}}, (x,y) \in ([x_0,M] \cap T) \times ([y_0,N] \cap T) \tag{3}$$

provided that $B_5 < 1$, where

$$\overline{C}=C+\int_{y_0}^{N}\int_{x_0}^{M}[f_2(s,t)\frac{p-q}{p}K^{\frac{q}{p}}+g_2(s,t)\frac{p-r}{p}K^{\frac{r}{p}}+\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t$$

$$B_1(x,y)=\int_{y_0}^{y}\int_{x_0}^{x}[f_1(s,t)\frac{p-q}{p}K^{\frac{q}{p}}+g_1(s,t)\frac{p-r}{p}K^{\frac{r}{p}}+\int_{y_0}^{t}\int_{x_0}^{s}h_1(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t$$

$$B_2(x,y)=\int_{x_0}^{x}[f_1(s,y)\frac{q}{p}K^{\frac{q-p}{p}}+g_1(s,y)\frac{r}{p}K^{\frac{r-p}{p}}+\int_{y_0}^{y}\int_{x_0}^{s}h_1(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\Delta\xi\Delta\eta]\Delta s\cdot$$

$$B_3(x,y)=1+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(x,t)\Delta t$$

$$B_4(x,y)=B_1(x,y)+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(x,t)B_1(x,t)\Delta t$$

$$B_5=\int_{y_0}^{y}\int_{x_0}^{x}[f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}B_3(s,t)+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}B_3(s,t)]\Delta s\Delta t+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}B_3(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t\cdot$$

$$B_6=\int_{y_0}^{y}\int_{x_0}^{x}[f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}B_4(s,t)+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}B_4(s,t)]\Delta s\Delta t+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}B_4(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t\cdot$$

Proof : Let the right side of (1) be $v(x,y)$. Then

$$u(x,y)\le v^{\frac{1}{p}}(x,y),(x,y)\in([x_0,M]\cap T)\times([y_0,N]\cap T) \tag{4}$$

From (2) we have

$$u(\tau_1(x),\tau_2(y))\le v^{\frac{1}{p}}(x,y),(x,y)\in([x_0,M]\cap T)\times([y_0,N]\cap T) \tag{5}$$

Given a fixed $X\in[x_0,M]\cap T$, and $x\in[x_0,X]\cap T, y\in[y_0,N]\cap T$, then

$$v(x,y)\le v(X,y),x\in[x_0,X]\cap T,y\in[y_0,N]\cap T \tag{6}$$

Furthermore, considering

$$v(x_0,y_0)=C+\int_{y_0}^{N}\int_{x_0}^{M}[f_1(s,t)u^q(\tau_1(s),\tau_2(t))+g_1(s,t)u^r(\tau_1(s),\tau_2(t))]\Delta s\Delta t+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_1(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t, \tag{7}$$

we have

$$\begin{aligned}v(X,y)=C&+\int_{y_0}^{y}\int_{x_0}^{X}[f_1(s,t)u^q(\tau_1(s),\tau_2(t))+g_1(s,t)u^r(\tau_1(s),\tau_2(t))]\Delta s\Delta t+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h_1(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t\\&+\int_{y_0}^{N}\int_{x_0}^{M}[f_2(s,t)u^q(\tau_1(s),\tau_2(t))+g_2(s,t)u^r(\tau_1(s),\tau_2(t))]\Delta s\Delta t+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t\\&\le C+\int_{y_0}^{y}\int_{x_0}^{X}[f_1(s,t)v^{\frac{q}{p}}(s,t)+g_1(s,t)v^{\frac{r}{p}}(s,t)]\Delta s\Delta t+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h_1(\xi,\eta)v^{\frac{m}{p}}(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t\end{aligned}$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}[f_2(s,t)u^q(\tau_1(s),\tau_2(t))+g_2(s,t)u^r(\tau_1(s),\tau_2(t))]\Delta s\Delta t+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t$$

$$=v(x_0,y_0)+\int_{y_0}^{y}\int_{x_0}^{X}[f_1(s,t)v^{\frac{q}{p}}(s,t)+g_1(s,t)v^{\frac{r}{p}}(s,t)]\Delta s\Delta t+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h_1(\xi,\eta)v^{\frac{m}{p}}(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t\cdot \tag{8}$$

Then a suitable application of Lemma 2.1 and Lemma 2.2 yields

$$v(X,y)\le v(x_0,y_0)+B_1(X,y)+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(X,t)(v(x_0,y_0)+B_1(X,t))\Delta t$$

$$=v(x_0,y_0)[1+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(X,t)\Delta t]+B_1(X,y)+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(X,t)(v(x_0,y_0)+B_1(X,t))\Delta t,y\in[y_0,N]\cap T\cdot \tag{9}$$

Combining (6), (9), it follows

$$v(x,y)\le v(x_0,y_0)+B_1(X,y)+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(X,t)(v(x_0,y_0)+B_1(X,t))\Delta t$$

$$=v(x_0,y_0)[1+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(X,t)\Delta t]+B_1(X,y)+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(X,t)(v(x_0,y_0)+B_1(X,t))\Delta t\quad x\in[x_0,X]\cap T,y\in[y_0,N]\cap T \tag{10}$$

Setting $x=X$ in (10), considering X is selected from $[x_0,M]\cap T$ arbitrarily, substituting X with x, yields

$$v(x,y)\le v(x_0,y_0)+B_1(x,y)+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(x,t)(v(x_0,y_0)+B_1(x,t))\Delta t$$

$$=v(x_0,y_0)[1+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(x,t)\Delta t]+B_1(X,y)+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(x,t)(v(x_0,y_0)+B_1(x,t))\Delta t,x\in[x_0,X]\cap T,y\in[y_0,N]\cap T \tag{11}$$

that is,

$$v(x,y)\le v(x_0,y_0)B_3(x,y)+B_4(x,y),x\in[x_0,X]\cap T,y\in[y_0,N]\cap T \tag{12}$$

On the other hand, from Lemma 2.2, (5) and (7) we obtain

$$v(x_0,y_0)\le C+\int_{y_0}^{N}\int_{x_0}^{M}[f_2(s,t)v^{\frac{q}{p}}(s,t)+g_2(s,t)v^{\frac{r}{p}}(s,t)]\Delta s\Delta t+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)v^{\frac{m}{p}}(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t$$

$$\le C+\int_{y_0}^{N}\int_{x_0}^{M}[f_2(s,t)(\frac{q}{p}K^{\frac{q-p}{p}}v(s,t)+\frac{p-q}{p}K^{\frac{q}{p}})$$

$$+g_2(s,t)(\frac{r}{p}K^{\frac{r-p}{p}}v(s,t)+\frac{p-r}{p}K^{\frac{r}{p}})]\Delta s\Delta t+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)(\frac{m}{p}K^{\frac{m-p}{p}}v(\xi,\eta)+\frac{p-m}{p}K^{\frac{m}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$

$$=\overline{C}+\int_{y_0}^{N}\int_{x_0}^{M}[f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}v(s,t)+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}v(s,t)]\Delta s\Delta t+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}v(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t \tag{13}$$

Then using (12) in (13) yields

$$v(x_0,y_0)\le \overline{C}+\int_{y_0}^{N}\int_{x_0}^{M}\{f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}[v(x_0,y_0)B_3(s,t)+B_4(s,t)]\}\Delta s\Delta t+\int_{y_0}^{N}\int_{x_0}^{M}\{g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}[v(x_0,y_0)B_3(s,t)+B_4(s,t)]\}\Delta s\Delta t$$

$$+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}v[v(x_0,y_0)B_3(\xi,\eta)+B_4(\xi,\eta)]\Delta\xi\Delta\eta\Delta s\Delta t$$

$$=\overline{C}+v(x_0,y_0)\{\int_{y_0}^{y}\int_{x_0}^{x}[f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}B_3(s,t)+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}B_3(s,t)]\Delta s\Delta t+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}B_3(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t\}$$

$$+\int_{y_0}^{y}\int_{x_0}^{x}[f_2(s,t)\frac{q}{p}K^{\frac{q-p}{p}}B_4(s,t)+g_2(s,t)\frac{r}{p}K^{\frac{r-p}{p}}B_4(s,t)]\Delta s\Delta t+\int_{y_0}^{N}\int_{x_0}^{M}\int_{y_0}^{t}\int_{x_0}^{s}h_2(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}B_4(\xi,\eta)\Delta\xi\Delta\eta]\Delta s\Delta t$$

$$=\overline{C}+v(x_0,y_0)B_5+B_6 \tag{14}$$

which is followed by

$$v(x_0,y_0)\le\frac{\overline{C}+B_6}{1-B_5} \tag{15}$$

Combining (4), (12) and (15) we can obtain the desired inequality (3).

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