

Analysis of a Nonlinear Integral Inequality on Time Scales

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Abstract. In this paper, a new Volterra-Fredholm type delay integral inequality on time scales is established, which can be used as a hand tool in the investigation of qualitative properties of delay dynamic equations.

Keywords: Delay integral inequality; Time scales; Dynamic equation; Bounded

1. Introduction

During the past decades, many integral inequalities have been established since then, for example [1-10], which have played an important role in the research of qualitative properties of solutions of dynamic equations.

Our aim in this paper is to establish a new Volterra-Fredholm type delay integral inequality on time scales, which provides new bound for unknown functions.

In the rest of the paper, R denotes the set of real numbers and $R_+ = [0, \infty)$. T denotes an arbitrary time scale and $T_0 = [x_0, \infty) \cap T, \bar{T}_0 = [y_0, \infty) \cap T$, where $x_0, y_0 \in T$. The set T^κ is defined to be T if T does not have a left-scattered maximum, otherwise it is T without the left-scattered maximum. On T we define the forward and backward jump operators $\sigma \in (T, T)$ and $\rho \in (T, T)$ such that $\sigma(t) = \inf\{s \in T, s > t\}$, $\rho(t) = \sup\{s \in T, s < t\}$.

Definition 1: A point $t \in T$ with $t > \inf T$ is said to be left-dense if $\rho(t) = t$ and right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$.

Definition 2: A function $f \in (T, R)$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1 + \mu(t)f(t) \neq 0$, where $\mu(t) = \sigma(t) - t$. C_{rd} denotes the set of rd-continuous functions, while \bar{R} denotes the set of all regressive and rd-continuous functions, and $\bar{R}^+ = \{f \mid f \in \bar{R}, 1 + \mu(t)f(t) > 0, \forall t \in T\}$.

Definition 3: For some $t \in T^\kappa$, and a function $f \in (T, R)$, the delta derivative of f is denoted by $f^\Delta(t)$, and satisfies $|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$ for $\forall \varepsilon > 0$, where $s \in U$, and U is a neighborhood of t . The function f is called delta differential on T^κ .

Similarly, for some $y \in T^\kappa$, and a function $f \in (T \times T, R)$, the partial delta derivative of f with respect to y is denoted by $(f(x, y))_y^\Delta$, and satisfies

$|f(x, \sigma(y)) - f(x, s) - (f(x, y))_y^\Delta(\sigma(y) - s)| \leq \varepsilon |\sigma(y) - s|$ for $\forall \varepsilon > 0$, where $s \in U$, and U is a neighborhood of y .

Definition 4: For some $a, b \in T$ and a function $f \in (T, R)$, the Cauchy integral of f is defined by

$$\int_a^b f(t) \Delta t = F(b) - F(a), \text{ where } F^\Delta(t) = f(t), t \in T^\kappa.$$

Similarly, for some $a, b \in T$ and a function $f \in (T \times T, R)$, the Cauchy partial integral of f with respect to

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y is defined by $\int_a^b f(x, y) \Delta y = F(x, b) - F(x, a)$, where $(F(x, y))_y^\Delta = f(x, y), y \in T^\kappa$.

More details on time scales can be referred to [11].

2. Main Results

We will give some lemmas for further use.

Lemma 2.1 ([11], Gronwall's inequality): Suppose $X \in T_0$ is an arbitrarily fixed number, and $u(X, y), b(X, y) \in C_{rd}$, $m(X, y) \in \overline{R}^+$ with respect to y , $m(X, y) \geq 0$, then

$$u(X, y) \leq b(X, y) + \int_{y_0}^y m(X, t) u(X, t) \Delta t, y \in \overline{T}_0$$

implies

$$u(X, y) \leq b(X, y) + \int_{y_0}^y e_m(y, \sigma(t)) b(X, t) m(X, t) \Delta t, y \in \overline{T}_0,$$

where $e_m(y, y_0)$ is the unique solution of the following equation

$$(z(X, y))_y^\Delta = m(X, y) z(X, y), z(X, y_0) = 1.$$

Lemma 2.2 [15]: Assume that $a \geq 0, p \geq q \geq 0$, and $p \neq 0$, then for any $K > 0$,

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}.$$

Theorem 2.1: Suppose $u, f_i, g_i, h_i \in C_{rd}(T_0 \times \overline{T}_0, R_+), i = 1, 2$, p, q, r, m, C, m, C are constants, and $p \geq q \geq 0, p \geq r \geq 0, p \geq m \geq 0, p \neq 0, C > 0, \tau_1 \in (T_0, T), \tau_1(x) \leq x, -\infty < \alpha = \inf\{\tau_1(x), x \in T_0\} \leq x_0, \tau_2 \in (\overline{T}_0, T), \tau_2(y) \leq y, -\infty < \beta = \inf\{\tau_2(y), y \in \overline{T}_0\} \leq y_0, \phi \in C_{rd}([\alpha, x_0] \times [\beta, y_0]) \cap T^2, R_+, M \in T_0, N \in \overline{T}_0$ are two fixed numbers, $K > 0$ is an arbitrary constant. If for $(x, y) \in ([x_0, M] \cap T) \times ([y_0, N] \cap T)$, $u(x, y)$ satisfies the following inequality

$$\begin{aligned} u^p(x, y) \leq & C + \int_{y_0}^y \int_{x_0}^x [f_1(s, t) u^q(\tau_1(s), \tau_2(t)) + g_1(s, t) u^r(\tau_1(s), \tau_2(t))] \Delta s \Delta t + \int_{y_0}^y \int_{x_0}^x \int_{y_0}^t \int_{x_0}^s h_1(\xi, \eta) u^m(\tau_1(\xi), \tau_2(\eta)) \Delta \xi \Delta \eta \Delta s \Delta t \\ & + \int_{y_0}^N \int_{x_0}^M [f_2(s, t) u^q(\tau_1(s), \tau_2(t)) + g_2(s, t) u^r(\tau_1(s), \tau_2(t))] \Delta s \Delta t + \int_{y_0}^N \int_{x_0}^M \int_{y_0}^t \int_{x_0}^s h_2(\xi, \eta) u^m(\tau_1(\xi), \tau_2(\eta)) \Delta \xi \Delta \eta \Delta s \Delta t \end{aligned} \quad (1)$$

with the initial condition

$$\begin{cases} u(x, y) = \phi(x, y), x \in [\alpha, x_0] \cap T, \text{ or } y \in [\beta, y_0] \cap T \\ \phi(\tau_1(x), \tau_2(y)) \leq C^{\frac{1}{p}}, \tau_1(x) \leq x_0, \text{ or } \tau_2(y) \leq y_0 \end{cases}, \quad (2)$$

then

$$u(x, y) \leq \left\{ \left[\frac{\overline{C} + B_6}{1 - B_5} B_3(x, y) + B_4(x, y) \right]^{\frac{1}{p}}, (x, y) \in ([x_0, M] \cap T) \times ([y_0, N] \cap T) \right. \quad (3)$$

provided that $B_5 < 1$, where

$$\bar{C} = C + \int_{y_0}^y \int_{x_0}^x [f_2(s,t) \frac{p-q}{p} K^{\frac{q}{p}} + g_2(s,t) \frac{p-r}{p} K^{\frac{r}{p}} + \int_{y_0}^t \int_{x_0}^s h_2(\xi,\eta) \frac{p-m}{p} K^{\frac{m}{p}} \Delta\xi\Delta\eta] \Delta s \Delta t$$

$$B_1(x,y) = \int_{y_0}^y \int_{x_0}^x [f_1(s,t) \frac{p-q}{p} K^{\frac{q}{p}} + g_1(s,t) \frac{p-r}{p} K^{\frac{r}{p}} + \int_{y_0}^t \int_{x_0}^s h_1(\xi,\eta) \frac{p-m}{p} K^{\frac{m}{p}} \Delta\xi\Delta\eta] \Delta s \Delta t$$

$$B_2(x,y) = \int_{x_0}^x [f_1(s,y) \frac{q}{p} K^{\frac{q-p}{p}} + g_1(s,y) \frac{r}{p} K^{\frac{r-p}{p}} + \int_{y_0}^y \int_{x_0}^s h_1(\xi,\eta) \frac{m}{p} K^{\frac{m-p}{p}} \Delta\xi\Delta\eta] \Delta s$$

$$B_3(x,y) = 1 + \int_{y_0}^y e_{B_2}(y, \sigma(t)) B_2(x,t) \Delta t$$

$$B_4(x,y) = B_1(x,y) + \int_{y_0}^y e_{B_2}(y, \sigma(t)) B_2(x,t) B_1(x,t) \Delta t$$

$$B_5 = \int_{y_0}^y \int_{x_0}^x [f_2(s,t) \frac{q}{p} K^{\frac{q-p}{p}} B_3(s,t) + g_2(s,t) \frac{r}{p} K^{\frac{r-p}{p}} B_3(s,t)] \Delta s \Delta t + \int_{y_0}^y \int_{x_0}^t \int_{y_0}^s h_2(\xi,\eta) \frac{m}{p} K^{\frac{m-p}{p}} B_3(\xi,\eta) \Delta\xi\Delta\eta] \Delta s \Delta t$$

$$B_6 = \int_{y_0}^y \int_{x_0}^x [f_2(s,t) \frac{q}{p} K^{\frac{q-p}{p}} B_4(s,t) + g_2(s,t) \frac{r}{p} K^{\frac{r-p}{p}} B_4(s,t)] \Delta s \Delta t + \int_{y_0}^y \int_{x_0}^t \int_{y_0}^s h_2(\xi,\eta) \frac{m}{p} K^{\frac{m-p}{p}} B_4(\xi,\eta) \Delta\xi\Delta\eta] \Delta s \Delta t$$

Proof : Let the right side of (1) be $v(x,y)$. Then

$$u(x,y) \leq v^{\frac{1}{p}}(x,y), (x,y) \in ([x_0, M] \cap T) \times ([y_0, N] \cap T) \quad (4)$$

From (2) we have

$$u(\tau_1(x), \tau_2(y)) \leq v^{\frac{1}{p}}(x,y), (x,y) \in ([x_0, M] \cap T) \times ([y_0, N] \cap T) \quad (5)$$

Given a fixed $X \in [x_0, M] \cap T$, and $x \in [x_0, X] \cap T, y \in [y_0, N] \cap T$, then

$$v(x,y) \leq v(X,y), x \in [x_0, X] \cap T, y \in [y_0, N] \cap T \quad (6)$$

Furthermore, considering

$$v(x_0, y_0) = C + \int_{y_0}^y \int_{x_0}^x [f_1(s,t) u^q(\tau_1(s), \tau_2(t)) + g_1(s,t) u^r(\tau_1(s), \tau_2(t))] \Delta s \Delta t + \int_{y_0}^y \int_{x_0}^t \int_{y_0}^s h_1(\xi,\eta) u^m(\tau_1(\xi), \tau_2(\eta)) \Delta\xi\Delta\eta \Delta s \Delta t, \quad (7)$$

we have

$$\begin{aligned} v(X,y) &= C + \int_{y_0}^y \int_{x_0}^X [f_1(s,t) u^q(\tau_1(s), \tau_2(t)) + g_1(s,t) u^r(\tau_1(s), \tau_2(t))] \Delta s \Delta t + \int_{y_0}^y \int_{x_0}^t \int_{y_0}^s h_1(\xi,\eta) u^m(\tau_1(\xi), \tau_2(\eta)) \Delta\xi\Delta\eta \Delta s \Delta t \\ &\quad + \int_{y_0}^y \int_{x_0}^X [f_2(s,t) u^q(\tau_1(s), \tau_2(t)) + g_2(s,t) u^r(\tau_1(s), \tau_2(t))] \Delta s \Delta t + \int_{y_0}^y \int_{x_0}^t \int_{y_0}^s h_2(\xi,\eta) u^m(\tau_1(\xi), \tau_2(\eta)) \Delta\xi\Delta\eta \Delta s \Delta t \\ &\leq C + \int_{y_0}^y \int_{x_0}^X [f_1(s,t) v^{\frac{q}{p}}(s,t) + g_1(s,t) v^{\frac{r}{p}}(s,t)] \Delta s \Delta t + \int_{y_0}^y \int_{x_0}^t \int_{y_0}^s h_1(\xi,\eta) v^{\frac{m}{p}}(\xi,\eta) \Delta\xi\Delta\eta \Delta s \Delta t \end{aligned}$$

$$\begin{aligned}
& + \int_{y_0}^{NM} \int_{x_0} [f_2(s,t)u^q(\tau_1(s), \tau_2(t)) + g_2(s,t)u^r(\tau_1(s), \tau_2(t))] \Delta s \Delta t + \int_{y_0}^{NM} \int_{x_0} \int_{y_0}^t \int_{x_0}^s h_2(\xi, \eta) u^m(\tau_1(\xi), \tau_2(\eta)) \Delta \xi \Delta \eta \Delta s \Delta t \\
& = v(x_0, y_0) + \int_{y_0}^X \int_{x_0} [f_1(s,t)v^{\frac{q}{p}}(s,t) + g_1(s,t)v^{\frac{r}{p}}(s,t)] \Delta s \Delta t + \int_{y_0}^X \int_{y_0}^t \int_{x_0}^s h_1(\xi, \eta) v^{\frac{m}{p}}(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t. \tag{8}
\end{aligned}$$

Then a suitable application of Lemma 2.1 and Lemma 2.2 yields

$$\begin{aligned}
v(X, y) & \leq v(x_0, y_0) + B_1(X, y) + \int_{y_0}^y e_{B_2}(y, \sigma(t)) B_2(X, t) (v(x_0, y_0) + B_1(X, t)) \Delta t \\
& = v(x_0, y_0) [1 + \int_{y_0}^y e_{B_2}(y, \sigma(t)) B_2(X, t) \Delta t] + B_1(X, y) + \int_{y_0}^y e_{B_2}(y, \sigma(t)) B_2(X, t) (v(x_0, y_0) + B_1(X, t)) \Delta t, y \in [y_0, N] \cap T. \tag{9}
\end{aligned}$$

Combining (6), (9), it follows

$$\begin{aligned}
v(x, y) & \leq v(x_0, y_0) + B_1(X, y) + \int_{y_0}^y e_{B_2}(y, \sigma(t)) B_2(X, t) (v(x_0, y_0) + B_1(X, t)) \Delta t \\
& = v(x_0, y_0) [1 + \int_{y_0}^y e_{B_2}(y, \sigma(t)) B_2(X, t) \Delta t] + B_1(X, y) + \int_{y_0}^y e_{B_2}(y, \sigma(t)) B_2(X, t) (v(x_0, y_0) + B_1(X, t)) \Delta t \quad x \in [x_0, X] \cap T, y \in [y_0, N] \cap T \tag{10}
\end{aligned}$$

Setting $x = X$ in (10), considering X is selected from $[x_0, M] \cap T$ arbitrarily, substituting X with x , yields

$$\begin{aligned}
v(x, y) & \leq v(x_0, y_0) + B_1(x, y) + \int_{y_0}^y e_{B_2}(y, \sigma(t)) B_2(x, t) (v(x_0, y_0) + B_1(x, t)) \Delta t \\
& = v(x_0, y_0) [1 + \int_{y_0}^y e_{B_2}(y, \sigma(t)) B_2(x, t) \Delta t] + B_1(x, y) + \int_{y_0}^y e_{B_2}(y, \sigma(t)) B_2(x, t) (v(x_0, y_0) + B_1(x, t)) \Delta t, x \in [x_0, X] \cap T, y \in [y_0, N] \cap T \tag{11}
\end{aligned}$$

that is,

$$v(x, y) \leq v(x_0, y_0) B_3(x, y) + B_4(x, y), x \in [x_0, X] \cap T, y \in [y_0, N] \cap T \tag{12}$$

On the other hand, from Lemma 2.2, (5) and (7) we obtain

$$\begin{aligned}
v(x_0, y_0) & \leq C + \int_{y_0}^{NM} \int_{x_0} [f_2(s,t)v^{\frac{q}{p}}(s,t) + g_2(s,t)v^{\frac{r}{p}}(s,t)] \Delta s \Delta t + \int_{y_0}^{NM} \int_{x_0} \int_{y_0}^t \int_{x_0}^s h_2(\xi, \eta) v^{\frac{m}{p}}(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t \\
& \leq C + \int_{y_0}^{NM} \int_{x_0} [f_2(s,t) (\frac{q}{p} K^{\frac{q-p}{p}} v(s,t) + \frac{p-q}{p} K^{\frac{q}{p}}) \\
& \quad + g_2(s,t) (\frac{r}{p} K^{\frac{r-p}{p}} v(s,t) + \frac{p-r}{p} K^{\frac{r}{p}})] \Delta s \Delta t + \int_{y_0}^{NM} \int_{x_0} \int_{y_0}^t \int_{x_0}^s h_2(\xi, \eta) (\frac{m}{p} K^{\frac{m-p}{p}} v(\xi, \eta) + \frac{p-m}{p} K^{\frac{m}{p}}) \Delta \xi \Delta \eta \Delta s \Delta t \\
& = \bar{C} + \int_{y_0}^{NM} \int_{x_0} [f_2(s,t) \frac{q}{p} K^{\frac{q-p}{p}} v(s,t) + g_2(s,t) \frac{r}{p} K^{\frac{r-p}{p}} v(s,t)] \Delta s \Delta t + \int_{y_0}^{NM} \int_{x_0} \int_{y_0}^t \int_{x_0}^s h_2(\xi, \eta) \frac{m}{p} K^{\frac{m-p}{p}} v(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t \tag{13}
\end{aligned}$$

Then using (12) in (13) yields

$$\begin{aligned}
v(x_0, y_0) &\leq \bar{C} + \int_{y_0}^y \int_{x_0}^x \{f_2(s, t) \frac{q}{p} K^{\frac{q-p}{p}} [v(x_0, y_0) B_3(s, t) + B_4(s, t)]\} \Delta s \Delta t + \int_{y_0}^y \int_{x_0}^x \{g_2(s, t) \frac{r}{p} K^{\frac{r-p}{p}} [v(x_0, y_0) B_3(s, t) + B_4(s, t)]\} \Delta s \Delta t \\
&\quad + \int_{y_0}^y \int_{x_0}^x \int_{y_0}^t \int_{x_0}^s h_2(\xi, \eta) \frac{m}{p} K^{\frac{m-p}{p}} v [v(x_0, y_0) B_3(\xi, \eta) + B_4(\xi, \eta)] \Delta \xi \Delta \eta \Delta s \Delta t \\
&= \bar{C} + v(x_0, y_0) \left\{ \int_{y_0}^y \int_{x_0}^x [f_2(s, t) \frac{q}{p} K^{\frac{q-p}{p}} B_3(s, t) + g_2(s, t) \frac{r}{p} K^{\frac{r-p}{p}} B_3(s, t)] \Delta s \Delta t + \int_{y_0}^y \int_{x_0}^x \int_{y_0}^t \int_{x_0}^s h_2(\xi, \eta) \frac{m}{p} K^{\frac{m-p}{p}} B_3(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t \right\} \\
&\quad + \int_{y_0}^y \int_{x_0}^x [f_2(s, t) \frac{q}{p} K^{\frac{q-p}{p}} B_4(s, t) + g_2(s, t) \frac{r}{p} K^{\frac{r-p}{p}} B_4(s, t)] \Delta s \Delta t + \int_{y_0}^y \int_{x_0}^x \int_{y_0}^t \int_{x_0}^s h_2(\xi, \eta) \frac{m}{p} K^{\frac{m-p}{p}} B_4(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t \\
&= \bar{C} + v(x_0, y_0) B_5 + B_6
\end{aligned} \tag{14}$$

which is followed by

$$v(x_0, y_0) \leq \frac{\bar{C} + B_6}{1 - B_5} \tag{15}$$

Combining (4), (12) and (15) we can obtain the desired inequality (3).

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