

# Constrained model predictive control for a class of nonlinear systems

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**Abstract.** We propose a robust model predictive controller design method for discrete-time systems with sector bounded nonlinearity. The sector condition is expressed by polytopic convex combination. Sufficient conditions for the model predictive controller synthesis are given in terms of a finite horizon of linear matrix inequalities (LMIs) using Lyapunov method.

**Keywords:** Model predictive control, Sector bounded nonlinear systems, LMI.

## 1. Introduction

Model predictive control(MPC) scheme is very useful since it is possible to handle input constraints and to use the current measurement state at the optimization algorithm. But if the model is not accurate, the control technique does not guarantee the stability and performance. To date, the MPC problem has been dealt with various systems such as linear uncertain systems [1-4], and linear parameter varying systems [5-7]. In practice, all physical systems are nonlinear in nature and there are various kinds of nonlinearities. Especially, sector bounded nonlinearities are commonly encountered in reality such as saturation, quantization, backlash, deadzone and so on.

Thus in this paper, we consider a sector bounded nonlinear model to handle systems with more general nonlinearities than polytopic ones [3],[7] and propose a one horizon robust MPC method for sector bounded nonlinear systems based on an augmented terminal weighting matrix in terms of a finite horizon of linear matrix inequalities (LMIs) [10]. This terminal weighting matrix has generalized Lur'e type Lyapunov function structure. To reduce the design conservatism, the equality condition and inequality constraints are derived with convex representation of sector bounds of the nonlinear function. These extra freedoms provide a less conservative design condition. Finally, we demonstrate the effectiveness of the proposed approach using numerical examples.

## 2. Problem statement

Consider the following linear parameter varying discrete-time systems

$$\begin{aligned}x(k+1) &= Ax(k) + Ff(q(x(k))) + Bu(k), \\q(k) &= Cx(k)\end{aligned}\tag{1}$$

with input constraints

$$-\bar{u} \leq u(k) \leq \bar{u}, \text{ for all } k \in [0, \infty),\tag{2}$$

where  $x \in R^n$  is the state vector,  $u \in R^m$  is the control input and  $f(q(k))$  is memoryless time-invariant nonlinearity with sector and slope restriction such as

$$a_i \leq \frac{f_i(q)}{q_i} \leq b_i,\tag{3}$$

$$a'_i \leq \frac{df_i(q)}{dq_i} \triangleq f'(q) \leq b'_i,\tag{4}$$

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in which  $a_i, b_i,$  and  $a'_i, b'_i$  are lower/upper sector and slope bounds, respectively.

For the simplicity, let us define

$$\delta^i(x_i(k)) = \frac{f_i(q_i(k))}{q_i(k)}, \quad (5)$$

then

$$f(q(k)) = \Delta Cx(k), \quad (6)$$

where  $\Delta = \text{diag}\{\delta^1(x_1), \dots, \delta^m(x_m)\}$ .

To describe the convexity of the nonlinearity, define

$$\begin{aligned} \Delta_1 &\triangleq \text{diag}\{a_1, \dots, a_m\}, \quad \Delta_2 \triangleq \text{diag}\{b_1, \dots, b_m\}, \\ \bar{\Delta}_1 &\triangleq \text{diag}\{a'_1, \dots, a'_m\}, \quad \bar{\Delta}_2 \triangleq \text{diag}\{b'_1, \dots, b'_m\}, \end{aligned} \quad (7)$$

then function  $\Delta(q)$  and  $\bar{\Delta}(q)$  can be represented by:

$$\Delta(q) \in \text{Co}\{\Delta_1, \Delta_2\}, \quad \bar{\Delta}(q) \in \text{Co}\{\bar{\Delta}_1, \bar{\Delta}_2\}, \quad (8)$$

where Co denotes convex hull.

Define  $\nabla := \text{diag}\{\Delta(q), \bar{\Delta}(q)\}$  in the following set

$$\Phi := \left\{ \text{diag}(\Delta, \bar{\Delta}) \mid \Delta \in \text{Co}\{\Delta_1, \Delta_2\}, \bar{\Delta} \in \text{Co}\{\bar{\Delta}_1, \bar{\Delta}_2\} \right\}. \quad (9)$$

The goal of this paper is to design a stabilizing control  $u(k)$  for (1) by the model predictive control strategy. To find such a control, we consider the following performance index

$$\begin{aligned} J(k, k + \infty) &\triangleq \sum_{j=0}^{\infty} \{ x(k+j|k)^T Q x(k+j|k) \\ &\quad + u(k+j|k)^T R u(k+j|k) \} \end{aligned} \quad (10)$$

where  $Q > 0$ ,  $R > 0$ . In order to design a control law for the future sampling time, we consider one step predictive controller with following structure at each time  $k$ ,

$$\begin{cases} u(k+j|k) = u^*(k|k), & \text{for } j = 0 \\ u(k+j|k) = Kx(k+j|k), & \text{for } j \geq 1 \end{cases} \quad (11)$$

that robustly minimizes the performance index (10).

For the cost monotonicity, the following inequality condition should be satisfied as

$$\begin{aligned} &V(k+j+1|k) - V(k+j|k) < \\ &-[x(k+j|k)^T Q x(k+j|k) + u(k+j|k)^T R u(k+j|k)] \end{aligned} \quad (12)$$

where

$$\begin{aligned} V(\bar{x}(k|k)) &= x_a(k|k)^T P x_a(k|k) > 0, \\ x_a(k) &= \begin{bmatrix} x(k) \\ f(q(k)) \end{bmatrix}. \end{aligned}$$

By summing (12) from  $i = 1$  to  $i = \infty$ , we obtain:

$$-V(x(k|k)) \leq -J(k, k+1). \quad (13)$$

For simplicity, we define

$$\begin{aligned} J^*(k) = & \text{Minimize} \quad \max_{u(k), P(a(k))\Delta(k+j), j \geq 1} J(k, k+1) \\ & \text{subject to (12)}. \end{aligned} \quad (14)$$

Thus our goal is redefined to find a robust MPC for minimization of (14) that minimize an upper bound on the worst case of the cost function  $J(k, k+1)$ .

### 3. Main result

**Theorem 1** Consider the system (1) at time instant  $k$ . The state feedback MPC law that minimizes  $J^*(k, k + \infty)$  can be solved by the following semidefinite programming

$$\begin{aligned} & \text{Minimize} \quad \gamma(k) \\ & u(k), G(k), H(k), Q(k) \end{aligned} \quad (15)$$

subject to

$$\begin{bmatrix} \gamma(k) & * & * & * \\ \Omega(k) & Q(k) & * & * \\ Q^{1/2}x(k) & 0 & I & * \\ R^{1/2}u(k) & 0 & 0 & I \end{bmatrix} \geq 0 \quad (16)$$

$$\begin{bmatrix} \Phi_{11} & * & * & * & * \\ \Phi_{11} & \Phi_{11} & * & * & * \\ \Phi_{11} & \Phi_{11} & \Phi_{11} & * & * \\ \Phi_{11} & \Phi_{11} & 0 & -Q(k) & * \\ \Phi_{11} & 0 & 0 & 0 & -I \end{bmatrix} > 0 \quad (17)$$

$$\begin{bmatrix} -\nabla^T \\ I \end{bmatrix}^T \Pi \begin{bmatrix} -\nabla^T \\ I \end{bmatrix} \leq 0, \quad (18)$$

$$\begin{bmatrix} U(k) & H(k) \\ H^T(k) & -Q_{11}(k) \end{bmatrix} \geq 0, \quad U_{ii}(k) \leq \bar{u}_i^2, \quad (19)$$

where

$$\begin{aligned} \Omega(k) &= \begin{bmatrix} Ax(k|k) + Bu(k|k) + Ff(q(k)) \\ \Delta C(Ax(k|k) + Bu(k|k) + Ff(q(k))) \end{bmatrix}, \Phi_{11} = G(k) + G^T(k) - Q(k), \\ \Phi_{21} &= \Delta C[AX(k) + BH(k) \quad FY(k)], \Phi_{22} = -2M(k), \\ \Phi_{31} &= \begin{bmatrix} CX(k) & 0 \\ C(AX + BH(k) - X(k)) & CFY(k) \\ 0 & Y(k) \\ 0 & -Y(k) \end{bmatrix}, \Phi_{32} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ M \end{bmatrix}, \Phi_{33} = -\Pi, \\ \Phi_{41} &= \begin{bmatrix} AX(k) + BH(k) & FY(k) \\ 0 & 0 \end{bmatrix}, \Phi_{42} = \begin{bmatrix} FM(k) \\ 0 \end{bmatrix}, \Phi_{51} = \begin{bmatrix} Q^{1/2}X(k) & 0 \\ \mathcal{R}^{1/2}H(k) & 0 \end{bmatrix}, \\ H(k) &= K(k)X(k). \end{aligned}$$

Proof: Detailed proof is omitted due to space limitations.

**Remark 1.** If there exist a feasible solution of the problem in Theorem 1 at time  $k = 0$ , then the system (1) with the MPC is robustly asymptotically stable. Since  $J^*(k)$  is greater than or equal to zero and strictly decreases as time goes to infinity, it plays a role of a Lyapunov function. Therefore we conclude that the closed-loop system is asymptotically stable. Also, the optimal solution of the optimization problem at the time  $k$  can be the solution at the time  $k + 1$ , the feasible solution of the optimization problem at time  $k$  is also feasible at the next time  $k + 1$ .

## 4. Numerical Example

Consider the Rotational-Translational Actuator (RTAC) system [13] to show the effectiveness of the proposed method

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 + 0.1 \sin(z_3) \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= u. \end{aligned} \quad (20)$$

Using Euler's first order approximation for the derivative and a sampling time 0.1 sec, we obtain a discrete-time sector bounded system (1) with the following matrices.

$$A = \begin{bmatrix} 1 & 0.1 & 0 & 0 \\ -0.1 & 1 & 0.01 & 0 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B_u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.1 \end{bmatrix}, \quad (21)$$

$$B = \begin{bmatrix} 0 \\ -0.01 \\ 0 \\ 0 \end{bmatrix}, C = [0 \quad 0 \quad 1 \quad 0], D = 0,$$

$$\phi(z_3(k)) = z_3(k) - \sin(z_3(k)). \quad (22)$$

For the initial condition  $x_0 = [0.038 \quad 0 \quad 160\pi/180 \quad 0]^T$  and the weighting matrices  $Q = 10I, R = I$  under the input constraint  $\bar{u} = 3$ , Fig. 1. Shows that the proposed method stabilize well the system (21) within the range  $z_3(k) \in [-160\pi/180, 160\pi/180]$ .

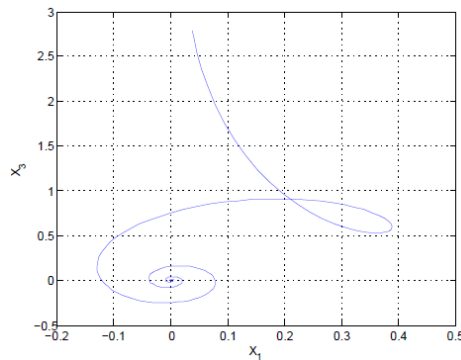


Figure 1: Simulation result (States trajectories)

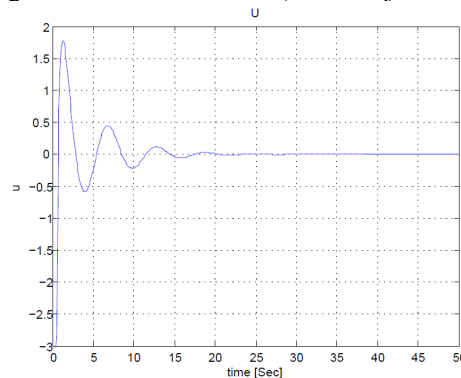


Figure 2: Simulation result (Control input)

## 5. Conclusion

In this paper, we proposed a new robust MPC algorithm for a discrete time system with sector nonlinearity. The cost monotonicity and one step predictive controller design conditions were expressed in the form of a finite number of LMIs. Numerical example shows the effectiveness of the proposed method.

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