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The Structure of Minimal Non-ST-Groups

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Abstract. A finite group G is called an ST-group if, for subgroups H, K and L with H s-seminormal in K and K s-seminormal in L, it is always the case that H is s-seminormal in L. In this paper, we classify these groups which are not ST-groups but whose proper subgroups are all ST-groups.

Keywords: minimal non-ST-group; s-seminormal subgroup; supersolvable group; power automorphism

1. Introduction

In this paper, only finite groups are considered. Our notation is standard and can be found in [5].

Let G be a group. A subgroup H of G is said to be permutable in G if HN = NH for every subgroup N of G. A subgroup H of G is said to be s-permutable in G if HS = SH for every Sylow subgroup S of G. A subgroup H of G is s-seminormal in G if H permutes with every Sylow p-subgroup of G with (p, |H|) = 1.

A group G is called a T-group (PT-group, PST-group, ST-group respectively) if, for subgroups H, K and L with H normal (permutable, s-permutable, s-seminormal respectively) in K and K normal (permutable, s-permutable, s-seminormal respectively) in L, it is always the case that H is normal (permutable, s-permutable, s-seminormal respectively) in L.

Gaschütz [4] determined the structure of solvable T-groups. Solvable PT-groups were studied and classified by Zacher [10]. The structure of solvable PST-groups was determined by Agrawal [1], see also Asaad and Csörgö [2]. The structure of solvable ST-groups were completed by Wang [8]. The further results can be found in [6].

In this paper, we call a group G a minimal non-ST-group if every proper subgroup of G is an ST-group but G itself is not, and the minimal non-ST-groups are classified.

2.Preliminary Results

We collect some lemmas which will be frequently used in the sequel.

Lemma 2.1: Let G be a group. Then the following results are equivalent:

(1) G is an ST-group.

(2) Every Sylow subgroup is s-seminormal in G, there exists an odd abelian normal Hall subgroup L of G such that G/L is a nilpotent group and the elements of G induce power automorphisms in L.

(3) Every subgroup of G of prime power order is s-seminormal in G.

Proof By [8, Theorem 1.1, Theorem 2.1, Corollary 2.1], we have easily prove it.

By [1] and Lemma 2.1, the following result is obvious.

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Lemma 2.2: Let G be a solvable ST-group. Then

(1) G is a solvable PST-group.

(2) G is supersolvable.

Lemma 2.3: [3] Let G be a minimal non-supersolvable group. Then

(1) G has only one normal Sylow p-subgroup P.

(2) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ and $P/\Phi(P)$ is non-cyclic.

(3) If $p \neq 2$, then the exponent of P is p.

(4) If P is non-abelian and $p = 2$, then the exponent of P is 4.

(5) If P is abelian, then the exponent of P is p.

(6) If P is non-abelian, then $\Phi(P) = P' = Z(P)$ is elementary abelian.

Lemma 2.4: Let G be a minimal non-ST-group. Then G is solvable, $|\pi(G)|$ is either 2 or 3, and G has a normal Sylow p-subgroup P.

Proof If G is not solvable, then there exists a non-solvable subgroup M of G such that every proper subgroup of M is solvable. By the hypothesis and Lemma 2.2, M is minimal non-supersolvable. Applying Lemma 2.3, M is solvable, a contradiction. Hence G is solvable.

Lemma 2.2 implies that G is either supersolvable or minimal non-supersolvable. If G is supersolvable, then G has a normal Sylow p-subgroup P. Applying Lemma 2.3, G has also a normal Sylow p-subgroup P if G is minimal non-supersolvable.

Since G is a minimal non-ST-group, applying Lemma 2.1, there exists a subgroup A of G with prime power order and a Sylow subgroup S of G such that $AS \neq SA$ where $(|A|, |S|) = 1$. By [8, Property B], we have $\langle x\rangle S \neq S\langle x\rangle$ where $x \in A$. The choice of G implies $G = \langle S, x\rangle$.

Clearly $P \neq S$. If $x \in P$, then $G = PS$ and so $|\pi(G)| = 2$. If x is not in P, then $G/P \cong H$ is an ST-group, where H is a p′-Hall subgroup of G. By Lemma 2.1,

$\langle x\rangle P/P \cdot SP/P = SP/P \cdot \langle x\rangle P/P$.

Hence

$G = \langle S, x\rangle = PS\langle x\rangle$.

Thus, $|\pi(G)| = 3$.

Lemma 2.5: [7, Lemma 5] Suppose $G = P\langle x\rangle$, P a normal Sylow p-subgroup of G and x is a q-element. If G is a group in which all maximal subgroups of P are normal, then there exists a positive integer l such that (aΦ(P))x = alΦ(P) for every element a of P, that is, x induces a power automorphism in $P/\Phi(P)$.

Lemma 2.6: [5, 13.4.3] Let α be a power automorphism of an abelian group A. If A is a p-group of finite exponent, then there is a positive integer l such that aα = al for all a in A. If α is nontrivial and has order prime to p, then α is fixed-point-free.

Lemma 2.7: [9, Theorem 2.8] Assume that G is a minimal non-nilpotent group. Then G is supersolvable if and only if the normal Sylow p-subgroup of G is cyclic.

3. Main Results

In this section, we give the classification of minimal non ST-groups.

Theorem 3.1: Let G be a minimal non-ST-group and $|\pi(G)| = 2$. Then there exists a normal Sylow p-subgroup P and a non-normal cyclic Sylow q-subgroup $Q = \langle y\rangle$ of G such that $G = PQ$, and G must be one of the following types:

(I) $P = \langle a, b\rangle$ is elementary abelian of order p2, ay = ak, by = b, where k - 1 is not divided by p and kq ≡ 1(mod p).

(II) P is abelian with $d(P) = 2$, y induces two different fixed-point-free power automorphisms in R and K respectively, yq induces a power automorphism in P, where R and K are only maximal subgroups of P which are normal in G, d(P) is the rank of P.

(III) P is minimal non-abelian with d(P) = 2, y induces two different fixed-point-free power automorphisms of order q in R and K respectively, yq ∈ CG(P), where R and K are only maximal subgroups of P which are normal in G.

(IV) G is minimal non-nilpotent with P is non-cyclic.

(V) G is minimal non-supersolvable, P is elementary abelian and yq induces a nontrivial power automorphism in P.

(VI) G is minimal non-supersolvable, where Φ(P) = P ′ = Z(P) is elementary abelian, yq ∈ CG(P) and y induces a nontrivial power automorphism in Φ(P).

Conversely, if a group G has one of the above presentations, then G must be a minimal non-ST-group.

Proof If G is a minimal non-ST-group and |π(G)| = 2, then we may assume G = PQ by Lemma 2.4, where P is a normal Sylow p-subgroup of G, Q is a Sylow q-subgroup of G and Q is not normal in G. Suppose that Q is non-cyclic, we have that PQ1 and PQ2 are both ST-groups by the hypothesis for two maximal subgroups Q1 and Q2 of Q. By Lemma 2.1, it is easy to see that G is an ST-group, a contradiction. Thus, Q is cyclic and we let Q = <y>. Since all the Sylow q-subgroups are conjugate in G, we only consider the case that y acts on P.

(1) Assume that G is supersolvable and d(P) = k where d(P) is the rank of P.

Choose a chief series of G 1 ◁ ... ◁ R ◁ P ◁ ... ◁ G.

By Maschke's Theorem [5, 8.1.2], there exists a subgroup N of P such that

P/Φ(P) = R/Φ(P) × N/Φ(P) where |N/Φ(P)| = p and N/Φ(P) ◁ G/Φ(P). Clearly N ◁ G, N ⩽̸ R, and 1 ◁ N ◁ P ◁ G is a normal series of G. Applying Schreier's Refinement Theorem [5, 3.1.2], P has a maximal subgroup K such that K is normal in G and K ≠ R. Therefore, P has at least two maximal subgroups R and K which are normal in G. Now we prove k = 2. If k ≥ 3, then we can let P/Φ(P) = <a1Φ(P)> × <a2Φ(P)> × ... × <akΦ(P)> where a1, a2, ... , ak-1 ∈ R, a2, a3, ... , ak ∈ K. As R<y> is an ST-group, we have that every maximal subgroup of R is normal in R<y>. It follows from Lemma 2.5 that (rΦ(R))y = rlΦ(R) for every element r of R, where l is a positive integer. Thus, (rΦ(P))y = rlΦ(P) for every element r of R. Similarly, (kΦ(P))y = kmΦ(P) for every element k of K, where m is a positive integer. Hence a2lΦ(P) = (a2Φ(P))y = a2mΦ(P) and so l ≡ m (mod p). Thus, (aiΦ(P)) y = ailΦ(P) for i = 1, 2, ... , k. It is easy to see that y induces a power automorphism in P/Φ(P). Lemma 2.5 implies that every maximal subgroup of P is normal in G. By induction and Lemma 2.1, we have easily that G is an ST-group. This contradiction implies k = 2.

Now we let P/Φ(P) = R/Φ(P) × K/Φ(P) = <a1Φ(P)> × <a2Φ(P)>, where a1 ∈ R, a2 ∈ K, and <a1Φ(P)>y = a1rΦ(P), <a2Φ(P)>y = a2sΦ(P). If r ≡ s (mod p), then every maximal subgroup P is normal in G. By induction, G is an ST-group, a contradiction. Hence r - s is not divided by p. Furthermore, we have that P has only two maximal subgroups which are normal in G. Clearly, at least one action of which y acts on R or K is nontrivial. Without loss of generality, we may assume that y induces a nontrivial automorphism α of order qi in R for some positive integer i. Since R<y> is an ST-group, it follows that every subgroup of R is normal in R<y> from Lemma 2.1, where R is abelian. By Lemma 2.6, α is fixed-point-free. So we have either K ∩ R = 1 if K<y> = K × <y> or K<y> ≠ K × <y>. If K ∩ R = 1 and K<y> = K × <y>, then P is an elementary abelian group of order p2 and so G is of type (I). If K<y> ≠ K × <y>, similar arguments as above, K is abelian and y induces a fixed-point-free power automorphism in K. Thus, Φ(P) = R ∩ K ≤ Z(P).

If P = Z(P), then P is abelian, and yq induces a power automorphism in P. So G is of type (II). If |P : Z(P)| = p, then P is abelian, a contradiction. If |P : Z(P)| = p2, then Φ(P) = R ∩ K = Z(P), and so P is minimal non-abelian. If [P, yq] ≠ 1, then by Lemma 2.1, P is abelian, a contradiction. Hence [P, yq] = 1. So G is of type (III).

(2) Assume that G is minimal non-supersolvable. By Lemma 2.3, P ia a special p-group (A p-group G is called a special p-group if either G is elementary abelian or Φ(P) = P′ = Z(P) is elementary abelian).

Now we prove that if Q ≤ M and M is a maximal subgroup of G, then Φ(P) is a Sylow p-subgroup of M.

Denote M =P3Q, where P3 is a Sylow subgroup of M. By [P3, Q] ≤ P ∩ P3Q = P3, NG(P3) ≥ P3Q = M. And since NG(P3) > P3, we have that P3 is normal in G. By Lemma 2.3, P3 ≤ Φ(P). The maximality of M implies that P3 = Φ(P) is a Sylow subgroup of M.

Case 1 Lemma 2.1 and Lemma 2.3 imply that a minimal non-nilpotent group which is non-supersolvable is a minimal non-ST-group. So G is of type (IV).

Case 2 If G is not a minimal non-nilpotent group and P is abelian, then for an ST-subgroup P<yq> of G, by Lemma 2.1, yq induces a nontrivial power automorphism in P. So G is of type (V).

Case 3 Assume that G is not a minimal non-nilpotent group and P is a non-abelian special p-group. For a maximal subgroup P<yq> of G, if P<yq> ≠ P × <yq>, then by Lemma 2.1, P is abelian, a contradiction. So P<yq> = P × <yq>, Φ(P)<y> ≠ Φ(P) × <y>.

By Lemma 2.1, y induces a nontrivial power automorphism in Φ(P). So G is of type (VI).

Conversely, by [8, Corollary 2.2], all subgroups of an ST-group are ST-groups. So we only prove that every maximal subgroup of a minimal non-ST-group is an ST-group.

Clearly, G of type (I) is a minimal non-ST-group.

If G is of type (II), then G has only maximal subgroups R<y>u, K<y>u and P<yq>u where u∈G. It is easy to see that G is a minimal non-ST-group.

If G is of type (III), then by the similar arguments as above, G has only maximal subgroups R<y>u, K<y>u and P<yq>u where u∈G. Clearly, G is a minimal non-ST-group.

For type (IV), it follows that G is non-supersolvable from Lemma 2.7. Lemma 2.1 implies that G is not an ST-group. So G is a minimal non-ST-group.

For type (V), in the same way as above, G is not an ST-group and G has maximal subgroups P<yq>u and Qu where u∈G. Since yq induces a power automorphism in P, we have that every subgroup of P<yq>u is an ST-group by Lemma 2.1. So G is a minimal non-ST-group.

For type (VI), by Lemma 2.1, G is not an ST-group and G has maximal subgroups P<yq>u and Φ(P)Qu, where u∈G. Since yq∈CG(P) and y induces a power automorphism in Φ(P), it follows that G is a minimal non-ST-group from Lemma 2.1. □

Theorem 3.2: Let G be a minimal non-ST-group and |π(G)| = 3. Then G = P(Q × R), where P is an abelian normal Sylow p-subgroup of G, Q and R are not normal in G, Q∈Sylq(G), R∈Sylr(G) and p > q > r, every element of G induces a power automorphism in P.

Proof Since all proper subgroups are ST-groups, it follows that all proper subgroups are PST-groups from Lemma 2.2. If G is a minimal non-PST-group, then by [2, Theorem 3], |π(G)| = 2, a contradiction. So G is a PST-group. Now we may assume G = PQR, where P∈Sylp(G), Q∈Sylq(G), R∈Sylr(G) with p > q > r, and P is normal in G, Q is normalized by R. Applying [2, Theorem 2], L is an odd abelian normal Hall subgroup of G, and every element of G induces a power automorphism in L, where L is the nilpotent residual of G. If P < L, then we may asssume Q ≤ L. It is easy to see that G is an ST-group by Lemma 2.1, a contradiction. Thus P = L. Applying [2, Theorem 2] again, the rest is clear.

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