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Some sufficient and necessary conditions for nilpotent groups

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Abstract. In this paper, we first define a special subgroup (i.e quasi-c-normal subgoup)of a finite group G which is a generalization of normal subgroups. A subgroup H of a group G is said to be quasi-c-normal in G if these exists a normal subgroup K of G such that |G : HK| is a power of a prime and $H \cap K \leqslant H^G$. Then we obtain some sufficient and necessary conditions for finite nilpotent groups by means of the quasi-c-normality of some subgroups of a group G and some know results are hence generalized.

Keywords: Sylow subgroups; Minimal subgroups; Quasi-cnormal subgroups; Nilpotent groups.

1. INTRODUCTION

All groups considered in this paper are finite groups. Most of the notation is standard and can be found in [3] and [2].

In [4, III, Theorem 5.5], Ito proved that if G is a group of odd order and all minimal subgroups of G lie in the center of G, then G is nilpotent. After Ito, many authors extent this result in different ways. For example, it is proved in [4]that if for an odd prime p, every subgroup of G of order p lies in the center of G, then G is p-nilp-otent. If all the elements of G of order 2 and 4 lie in the center of G, then G is 2-nilpotent(see [4, IV, Theorem 5.5]). Buckly [1] proved that a group of odd order is supersolvable if every minimal subgroup of G is normal in G. Later Shaalan [5] proved that if G is a finite group and every subgroup of prime order or order 4 is quasino-rmal in G, thenGis supersolvable. Wang [6] proved that G is supersolvable if every subgroup of G of primeorder or order 4 is c-normal in G.

Definition 1.1: A subgroup H of a group G is said to be quasi-c-normal in G if these exists a normal subgroup K of G such that |G : HK| is a power of a prime and $H \cap K \leqslant H^G$,where H^G is the core of G.

In this paper, we obtain some new characterizations of nilpotent groups asexten-sions of Ito's result on the base of the quasi-c-normality.

Recall that a groupHis said to be c-normal in G if these exists a normal subgroup N of G such that HN = G and H $\cap$N $\leqslant$ HG(see [6]). A subgroup H of a group G is called c-supplemented in G if there exists a subgroup K of G such that G = HK and H $\cap$ K $\leqslant$ H_G (see [7]). It is clear that

Every c-normal subgroup is both c-supplemented subgroup and quasi-c-normal subgroup. The following two examples show that in general a quasi-c-normal subgroup may neither be a c-normal subgroup nor be a c-supplemented subgroup.

Example 1.1: Let G = PSL(2, 7), and H is a normalizer of some 7-Sylow subgroup of G. Then it is clear that H is quasi-c-normal in G but it is not c-normal in G.

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Example 1.2: Let G = Z3~<a> , the wreath product of a cyclic group Z3 and a cyclic group <a>, where o(a) = 4. Then $<a^2>$ is quasi-c-normal in G but it is not c-supplem-ented.

In the following we list one lemma which is useful in the sequel.

Lemma 1.1: Let H and K be subgroups of a group G withH ⩽ M, and let N be a normal subgroup of G.

(1) If H is quasi-c-normal in G, then H is quasi-c-normalin K.

(2)Suppose that N ◁ G and N ⩽ H. Then H is quasi-c-normal in G if and only if H/N is quasi-c-normal in quasi-c-normal in G/N.

(3) If H is quasi-c-normal in G with (|H|, |N|) = 1, then the subgroup HN/N is quasi-c-normal in G/N.

Proof. (1)Since H is quasi-c-normal in G by assumption, there is a normal subgroup K of G such that |G : HK| is a prime power and H ∩ K ⩽ H^G. Let L = M ∩ K, then we have L is normal in M and M ∩ HK = H(M ∩ K) = HL. Since H ∩ L = H ∩M ∩ K ⩽ HG ∩M = H_M ,we have |M : HL| = |M/ M∩ HK | = |G/HK | is a prime power. Hence H is quasi-c-normal in M.

(2) Suppose that H/N is quasi-c-normal in quasi-c-normal in G/N. Then by definition of the quasi-c-normality, there exists a normal subgroup K of G such that |G/N :(H/N)(K/N)| is a prime power and H/N ∩ K/N ⩽ $(H/N)_{G/N}$. Therefore we may have |G : HK|is a prime power and H ∩ K ⩽H^G .

(3)If H is quasi-c-normal in G, then there is a normal subgroup K of G such that |G : HK| is a prime power and H ∩ K ⩽H^G . Hence |G/N : (HN/N)(KN/N)| =|G : HKN|||G : HK| is a prime power. On the other hand, (HN ∩ K)H = HN ∩ HK ⩽ HN, (HN ∩K)N ⩽HNand (|N|, |H|) = 1. Then by [2, A Lemma 1.6(c)], we have HN ∩ K = (H ∩ K)(N ∩ K). Thus HN/N ∩ KN/N = (HN ∩ KN)/N = (HN ∩ K)N/N = (H ∩ K) (N ∩ K) N/N = (H ∩ K)N/N ⩽ $(H/N)_{G/N}$, That is HN/N is quasi-c-normal in G/N. The proof is complete.

Lemma 1.2: [4, III, 5.2, p. 281] Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then

(i) G has a normal Sylow p-subgroup P for some prime p and G = PQ; where Q is a non-normal cyclic q-subgroup for some prime q is not equal to p;

(ii) P/_(P) is a minimal normal subgroup of G/_)P);

(iii) If P is non-abelian and p is not equal to 2, then the exponent of Pis p;

(iv) If P is non-abelian and p = 2, then the exponent of Pis 4;

(v) If P is abelian, then P is of exponent p.

2. THE MAIN RESULTS

Theorem 2.1: Let G be a finite group and G^N be the nilpotent residual of G. Then G is nilpotent if and only if every subgroup of G with prime order lies in the hyper-center $Z_\infty(G)$ of G and every cyclic subgroup of G^N with order 4 is quasi-c-normal in G.

Proof. Suppose that G is nilpotent. Then G = $Z_\infty(G)$ and
G^N = 1. Hence the necessity of our theorem is true. It is remain to prove that the converse is true.

Assume that all subgroups of G of prime order lie in Z_∞ (G)and that every cyclic subgroup with order 4 in G^N is quasi-c-normal in G. We now prove that G is nilpotent. Let G bea counterexample of minimal order and M be any nontrivialsubgroup of G. Since the nilpotent residual M^N of M iscontained in $G^N \cap M$ and $Z_\infty(G) \cap M \leqslant Z_\infty(M)$, wehave clearly that the hypothesis is true for M.Thus G is not nilpotent but every nontrivial subgroup of G is nilpotent by the choice of G. By lemma 1.2, G has a normal Sylow p-subgroup P such that G = PQ, where $Q \in Syl_q(G)$ and $q \neq p$. If p is an odd prime, then exp (P) = p. If p = 2, then exp(P) $\leqslant$ 4. Moreover, $P = G' = G^N$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. By the hypothesis, if p is an odd prime, then we have P is contained in $Z_\infty(G)$. Hence $G = PQ = Z_\infty(G)Q$ is nilpotent, which contradicts the choice of G. So we may assume that p = 2. Note in this case exp(P) $\leqslant$ 4. Let A = <a> be a subgroup of order 4 in $G^N = P$, then there exists a normal subgroup K of G such that $|G : AK| = r^\alpha$ and $A \cap K \leqslant A_G$, where r is a prime. Assume first that A is not normal in G. Then $A_G = 1$ or $A_G = \langle a^2 \rangle$. In this case, if r = p, then the Sylowq-subgroup is normal in G, which implies that $G= P \times Q$ is nilpotent, a contradiction. Suppose that r = q and $r^\alpha \neq 1$. Let K_p be a Sylow p-subgroup of K, then K_p is normal in G and hence $K_p\Phi(P)/\Phi(P)$ is normal in G too. By the minimality of $P/\Phi(P)$, we have $K_p\Phi(P) = P$. Hence $K_p = P$, which contradicts that A is not normal in G. So we may assume that A is a normal subgroup of G. Therefore $N_G(A)/C_G(A) = G/C_G(A)$ is a 2-group and hence $A \leqslant Z_\infty(G)$, which implies that $P \leqslant Z_\infty(G)$. Now $G = PQ = Z_\infty(G)Q$ is nilpotent, a contradiction again.

Theorem 2.2: Let G be a finite group and N be a normal subgroup of G such that G/N is nilpotent. Suppose that every element of prim order of $F^*(N)$ lies in the hyper center $Z_\infty(G)$ of G. Then G is nilpotent if and only if every cyclic subgroup of order 4 of $F^*(N)$ is quasi-c-normal in G, where $F^*(N)$ is the Generalized Fitting subgroup of N.

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Proof. We need only to prove the ”if” part.

Assume that the theorem is not true, let G be a counter example of minimal order. Every proper normal subgroup of G is nilpotent. sgkLet M be a maximal normal subgroup of G, then

$M/M \cap N$ is nilpotent. Moreover, since $F^*(M \cap N)$ is contained in $F^*(N)$ and $Z_\infty(G) \cap M$ lies in $Z_\infty(M)$, we know M satisfiesthe hypothesis of our theorem. The choice of G implies that M is nilpotent.

(2) F(G) is the unique maximal normal subgroup of G.

(2)follows immediately from (1).

(3) G is solvable .

Indeed, if N < G, then G is solvable by (1) and the hypothesis. Thus we may assume that N = G. In this case, if $F^*(N) = F^*(G) = G$, then G is nilpotent by Theorem 2.1. Therefore we may assume further that $F^*(N) = F^*(G) < G$. In addition, we have that G/F(G) is a nonabelian simple group. Hence G ‘ = G since G' is not contained in F(G). Since F(G) is not the identity group, we may choose a minimal prime divisor of |F(G)| such that the Sylow p-subgroup Pof F(G)is a nontrivial normal subgroup of G.For every element x of order p or order 4 (when p = 2), there exists a normal subgroup K of G such that $|G : <x>K| = q^{\alpha}$,and $<x> \cap K = <x>_G$, where q is a prime. If $q \neq p$ and $q^{\alpha} \neq 1$, then PK is a nontrivial normal subgroup of Gand hence (1) implies that PK is nilpotent. Thus we obtain that G is solvable since G/PK is nilpotent. If q = p andq $^{\alpha} \neq 1$, then K is a nontrivial normal subgroup of G. Again by (1), K is nilpotent. Since $|G/K| = p^{\beta}q^{\alpha}$, we have G / K is solvable, and hence G is solvable too. If G = $<x>K$, we then have that K = G by the maximality of F(G).Hence $<x>$ is normal in G and so $G = G' \leqslant C_G(x)$, that is to say G fixes every element x of order p nd order 4 (when p = 2). Therefore $G/C_G(P)$ is a p-group. Since $G = G'$, we have that $C_G(P) = G$ and hence $P \leqslant Z(G)$. Let $F^*(G/P) = L/P$, then there exists a chief series $P \trianglelefteq \ldots \trianglelefteq L_1 \trianglelefteq L$ such that G induces an inner automorphism on every chief factor of this series. Since $P \leqslant Z(G)$, we can extent this series to the identity group. Therefore $L \leqslant F^*(G) = F(G)$. Let G = G/P, then for every prime divisor of $|F^*(G)|$, we Have r > p. Now by the hypothesis, every element x of prime order of $F^*(G)$ is contained in $Z_{\infty}(G)$, which implies that $x \in Z_{\infty}(G) = Z_{\infty}(G)/P$.Therefore G satisfies the hypothesis of our theorem and hence G/P is nilpotent by the choice of G, which leads to G is solvable.

By (3), G is solvable and so $G^N < G$. Therefore $G^N \leqslant F(G) \cap N \leqslant F(N)$. Applying Theorem 2.1, we can obtain that G is nilpotent, a contradiction. This contradiction completes the proof of this theorem.

3. ACKNOWLEDGMENTS

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