

Variable Forgetting Factor Recursive Total Least Squares Algorithm for FIR Adaptive filtering

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Abstract. This paper proposes a variable forgetting factor recursive total least squares (VFF-RTLS) algorithm to recursively compute the total least squares solution for adaptive finite impulse response (FIR) filtering. The forgetting factor of the VFF-RTLS algorithm is updated by minimizing the cost function (Rayleigh quotient). In simulations, the VFF-RTLS algorithm provided both fast tracking capability and small mean square deviation. Based on improved precision to estimate the FIR of an unknown system and adaptability to change in the system, the VFF-RTLS algorithm can be applied extensively in adaptive signal processing areas.

Keywords: Adaptive filtering, parameter estimation, finite impulse response, Rayleigh quotient, recursive least squares.

1. Introduction

The total least squares (TLS) method [1] can provide unbiased estimates of system parameters even if both input and output are corrupted by white noise. TLS solution can be obtained by computing the singular value decomposition (SVD) of an augmented data matrix or its covariance matrix, and is the generalized eigenvector associated with the smallest eigenvalue of the matrix [1], [2]. However, the number of multiplication operations of the SVD for an $N \times N$ matrix is generally $O(N^3)$, so real-time signal processing applications using TLS are limited. To efficiently compute the solution, several methods have been studied.

Some of these studies have determined by considering the Rayleigh quotient as a cost function, and these methods have been used to develop efficient recursive TLS algorithms [3], [4]. A new recursive total least squares (N-RTLS) algorithm [4] was proposed for adaptive FIR filtering; this algorithm uses fast computation of the fast gain vector (FGV) and adaptation minimization of the Rayleigh quotient in [3].

However, the N-RTLS algorithm has a constant forgetting factor (i.e., it uses the previous estimation result with a fixed ratio to represent next estimation), so it is not suitable for tracking time-varying parameters because its convergence is slow when the forgetting factor is close to one, whereas the mean square deviation (MSD) is large when the forgetting factor is small. Therefore, to yield precise estimation of the FIR of an unknown system in time-varying environments, the N-RTLS algorithm must be modified.

This paper proposes the variable forgetting factor recursive total least squares (VFF-RTLS) algorithm, which adapts the N-RTLS algorithm to use a varying the forgetting factor. The control of the forgetting factor is based on the minimization of the cost function (Rayleigh quotient). In simulations, the VFF-RTLS algorithm had both faster tracking capability and smaller MSD than the N-RTLS algorithms.

2. Variable Forgetting Factor Recursive Total Least Squares Algorithm

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2.1. System model

Consider a system identification model. The unknown system has a FIR vector $\mathbf{h} \in R^{M \times 1}$. In this system, both input and output are corrupted by white Gaussian noise. The adaptive filter is used to estimate the unknown system.

The desired output is represented as

$$d(t) = \mathbf{x}^T(t)\mathbf{h} + n_o(t), \quad (1)$$

where $\mathbf{x}(t) = [x(t), x(t-1), \dots, x(t-M+1)]^T$ denotes the noise-free input vector, and $n_o(t)$ is observation noise which is white Gaussian with zero mean, and variance $\sigma_o^2(t)$. The noisy input vector of the adaptive FIR filter is given by

$$\tilde{\mathbf{x}}(t) = \mathbf{x}(t) + \mathbf{n}_i(t), \quad (2)$$

where $\mathbf{n}_i(t) = [n_i(t), n_i(t-1), \dots, n_i(t-M+1)]^T$, and $\mathbf{n}_i(t)$ is white Gaussian noise with zero mean, and variance $\sigma_i^2(t)$. Then, an autocorrelation matrix of $\tilde{\mathbf{x}}(t)$ is defined as

$$\tilde{R} = E[\tilde{\mathbf{x}}(t)\tilde{\mathbf{x}}^T(t)] = R + \sigma_i^2 I, \quad (3)$$

where $R = E[\mathbf{x}(t)\mathbf{x}^T(t)]$ is the autocorrelation matrix of the noise-free input vector. The augmented input vector is defined as $\bar{\mathbf{x}}(t) = [\tilde{\mathbf{x}}(t), d(t)]^T \in R^{(M+1) \times 1}$, and its autocorrelation matrix can be represented as

$$\bar{R} = E[\bar{\mathbf{x}}(t)\bar{\mathbf{x}}^T(t)] = \begin{bmatrix} \tilde{R} & \mathbf{b} \\ \mathbf{b}^T & c \end{bmatrix}, \quad (4)$$

where $\mathbf{b} = E[\tilde{\mathbf{x}}(t)d(t)]$ and $c = E[d(t)d(t)]$.

2.2. Rayleigh quotient and N-RTLS algorithm

One method to solve the TLS problem is to find the vector which minimizes the following Rayleigh quotient [3].

$$J(\mathbf{w}) = \frac{[\mathbf{w}^T, -1]\bar{R}[\mathbf{w}^T, -1]^T}{[\mathbf{w}^T, -1]\bar{D}[\mathbf{w}^T, -1]^T}, \quad (6)$$

where $\mathbf{w} \in R^{M \times 1}$ is a weight of the adaptive filter and $\bar{D} = \text{diag}(1, \dots, 1, \beta) \in R^{M \times M}$ is a weighting matrix with $\beta = \sigma_o^2 / \sigma_i^2$. Then, the TLS solution is given by $\mathbf{w}_{TLS} = \arg \min_{\mathbf{w}} J(\mathbf{w})$.

Stochastic quantities can be replaced by time-averaged values with a forgetting factor $\mu (0 < \mu \leq 1)$ for sufficiently large t such as

$$\tilde{R}(t) = \mu R(t-1) + \tilde{\mathbf{x}}(t)\tilde{\mathbf{x}}(t)^T \quad (7)$$

$$\mathbf{b}(t) = \mu \mathbf{b}(t-1) + \tilde{\mathbf{x}}(t)d(t) \quad (8)$$

$$c(t) = \mu c(t-1) + d(t)d(t). \quad (9)$$

In the N-RTLS algorithm [4], the weight vector of the adaptive filter is updated by

$$\mathbf{w}(t) = \mathbf{w}(t-1) + \alpha(t)\tilde{\mathbf{x}}(t), \quad (10)$$

where $\alpha(t)$ can be determined by $\min_{\alpha(t)} J(\mathbf{w}(t))$.

Using equation (10), $\min_{\alpha(t)} J(\mathbf{w}(t))$ can be rewritten as $\min_{\alpha(t)} J(\mathbf{w}(t-1) + \alpha(t)\tilde{\mathbf{x}}(t))$.

To find $\min_{\alpha(t)} J(\mathbf{w}(t-1) + \alpha(t)\tilde{\mathbf{x}}(t))$, let the gradient of $J(\mathbf{w}(t-1) + \alpha(t)\tilde{\mathbf{x}}(t))$ with respect to $\alpha(t)$ be equal to zero. Then

$$\frac{\partial J(\mathbf{w}(t-1) + \alpha(t)\tilde{\mathbf{x}}(t))}{\partial \alpha(t)} = 0. \quad (11)$$

We obtain $\alpha(t)$ by solving equation (11).

2.3. Proposed variable forgetting factor recursive total least squares algorithm

To adjust μ in the N-RTLS algorithm, the steepest descent method was applied to minimize the Rayleigh quotient $J(\mathbf{w})$ with respect to μ as follows.

$$\mu(t) = \mu(t-1) + \gamma_1 \frac{\partial J(\mathbf{w})}{\partial \mu}, \quad (12)$$

where γ_1 is a tuning parameter.

Using equation (7), (9), we can show that equation (6) is equivalent to

$$J(\mathbf{w}) = \frac{\mathbf{w}(t)^T (\mu \tilde{\mathbf{R}}(t-1) + \tilde{\mathbf{x}}(t) \tilde{\mathbf{x}}^T(t)) \mathbf{w}(t) + \mu c(t-1) + d(t)d(t)}{\beta + \mathbf{w}(t)^T \mathbf{w}(t)}, \quad (13)$$

and its derivative is easily computed as

$$\frac{\partial J(\mathbf{w})}{\partial \mu} = \frac{\mathbf{w}(t)^T \tilde{\mathbf{R}}(t-1) \mathbf{w}(t) + c(t-1)}{\beta + \mathbf{w}(t)^T \mathbf{w}(t)}. \quad (14)$$

Finally, we obtain the update equation for the forgetting factor

$$\mu(t) = \mu(t-1) + \gamma_1 \left(\frac{\mathbf{w}(t)^T \tilde{\mathbf{R}}(t-1) \mathbf{w}(t) + c(t-1)}{\beta + \mathbf{w}(t)^T \mathbf{w}(t)} \right). \quad (15)$$

However, forgetting factor should be zero because past information disturbs precise estimation when unknown system is changed suddenly. When the FIR of the system \mathbf{h} is changed, $e(t)^2$ is considerably increased. If estimation of \mathbf{h} is given by $\mathbf{w}(t)$ ($\mathbf{h} \approx \mathbf{w}(t)$), then $e(t)$ can be represented as

$$\begin{aligned} e(t) &= d(t) - \tilde{\mathbf{x}}(t)^T \mathbf{w}(t) = \mathbf{x}(t)^T \mathbf{h} + n_o(t) - (\mathbf{x}(t) + \mathbf{n}_i(t))^T \mathbf{w}(t) \\ &\approx n_o - \mathbf{n}_i(t)^T \mathbf{h}. \end{aligned} \quad (16)$$

To decide whether system has changed or not, we need two assumptions: $\sigma_i^2 \leq E[\mathbf{x}(t)^T \mathbf{x}(t)]$; $\sigma_o^2 \leq E[(\mathbf{x}(t)^T \mathbf{h})^2]$. From these assumptions, $E[e^2(t)]$ can be represented as

$$E[e^2(t)] = \sigma_o^2 + \|\mathbf{h}\|^2 \sigma_i^2 = \sigma_o^2 + \|\mathbf{h}\|^2 E[\mathbf{x}(t)^T \mathbf{x}(t)], \quad (17)$$

where

$$\sigma_o^2 \leq E[(\mathbf{x}(t)^T \mathbf{h})^2] = \|\mathbf{h}\|^2 E[\mathbf{x}(t)^T \mathbf{x}(t)]. \quad (18)$$

Using (18) in (17) yields

$$E[e^2(t)] \leq 2\|\mathbf{h}\|^2 E[\mathbf{x}(t)^T \mathbf{x}(t)]. \quad (19)$$

From independency of $\mathbf{x}(t)$ and $\mathbf{n}_i(t)$,

$$E[\tilde{\mathbf{x}}(t)^T \tilde{\mathbf{x}}(t)] = E[\mathbf{x}(t)^T \mathbf{x}(t)] + M \cdot \sigma_i^2 \leq (1+M) \cdot E[\mathbf{x}(t)^T \mathbf{x}(t)]. \quad (20)$$

Substituting (20) into (19),

$$E[e^2(t)] \leq \frac{2}{1+M} \|\mathbf{h}\|^2 E[\tilde{\mathbf{x}}(t)^T \tilde{\mathbf{x}}(t)]. \quad (21)$$

If we use a sufficiently large value of γ_2 , (21) can be approximated as,

$$e^2(t) \leq \gamma_2 \|\mathbf{h}\|^2 \|\tilde{\mathbf{x}}(t)\|^2 \approx \gamma_2 \|\mathbf{w}(t)\|^2 \|\tilde{\mathbf{x}}(t)\|^2. \quad (22)$$

Therefore, we can decide that the system is not changed if $e^2(t)$ satisfies (22).

From (15) and (22), it is inferred that the method of variable forgetting factor based on selective update is given by

$$\mu(t) = \begin{cases} 0, & \text{if } e(t)^2 > \gamma_2 \|\mathbf{w}(t)\|^2 \|\tilde{\mathbf{x}}(t)\|^2 \\ \mu(t-1) + \gamma_1 \left(\frac{\mathbf{w}(t)^T \tilde{\mathbf{R}}(t-1) \mathbf{w}(t) + c(t-1)}{\beta + \mathbf{w}(t)^T \mathbf{w}(t)} \right), & \text{otherwise} \end{cases} \quad (23)$$

The forgetting factor is updated in the direction of the Rayleigh quotient's steepest descent when \mathbf{h} is maintained. Furthermore, forgetting factor value become zero when \mathbf{h} is changed.

The forgetting factor of the N-RTLS algorithm is fixed at 0.997. A fixed forgetting factor cannot provide low estimation error when the system has arrived at the steady state. However, this disadvantage can be overcome by using our variable forgetting factor algorithm.

2.4. Simulation in MATLAB

We used computer simulation to confirm that the VFF-RTLS algorithm provides more accurate estimation of \mathbf{h} than other algorithms. We compare of the VFF-RTLS algorithm, the N-RTLS algorithm and the conventional recursive least squares algorithm (RLS) [5] with respect to MSD. The MSD value can be computed as

$$MSD = E \left[\frac{\|\mathbf{h} - \mathbf{w}(t)\|^2}{\|\mathbf{h}\|^2} \right]. \quad (24)$$

The adaptive filter and the unknown channel were assumed to have the same tap-lengths (M). A FIR of unknown system \mathbf{h} was randomly generated. The input signals were obtained by generating a white, zero mean, Gaussian random sequence. The signal to noise ratio (SNR) was calculated as

$$SNR_i = 10 \log_{10} \left(\frac{\sum_t \mathbf{x}^2(t)}{\sum_t n_i^2(t)} \right), \quad (25)$$

for the input signal and

$$SNR_o = 10 \log_{10} \left(\frac{\sum_t (\mathbf{x}^T(t)\mathbf{h})^2}{\sum_t n_o^2(t)} \right), \quad (26)$$

for the observation.

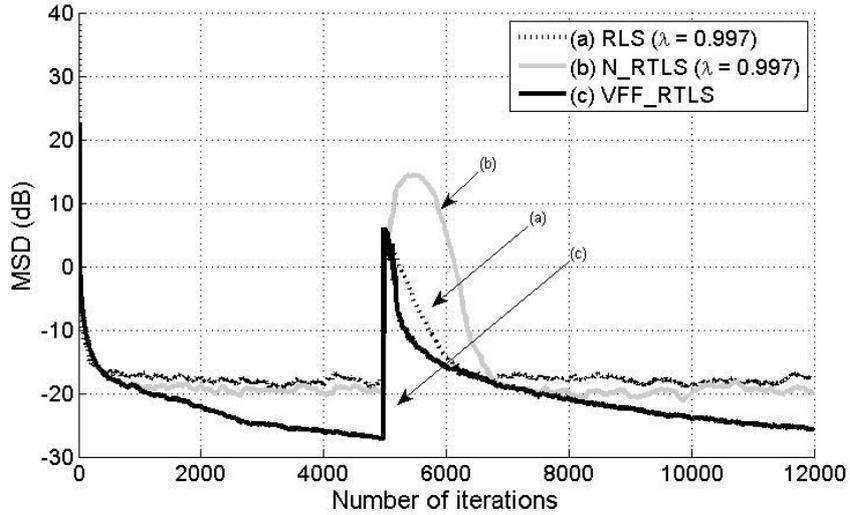


Fig.1. MSD curves of three algorithms for system identification simulation with white input sequence over 10 independent trials, where $SNR_i = 10 \text{ dB}$, $SNR_o = 10 \text{ dB}$, $M = 16$. (a) RLS and (b) N-RTLS with $\mu = 0.997$, (c) The proposed VFF-RTLS with $\gamma_1 = 0.01$, $\gamma_2 = 2$. The \mathbf{h} is changed at iterations 5000.

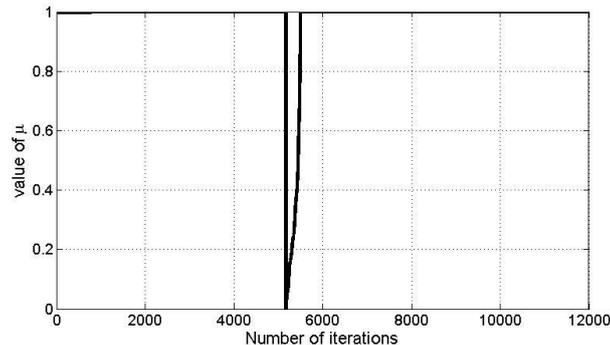


Fig.2. Evolution in time of the forgetting factor $\mu(t)$ of the VFF-RTLS

We computed MSD curves of the VFF-RTLS algorithm compared with the RLS, and the N-RTLS algorithm (Fig. 1). MSD was larger when using the RLS algorithm than when using the other two algorithms, because the system has a noisy input sequence (TLS problem). The MSD of the VFF-RTLS was less than that of the N-RTLS after 1000 iterations. Although the MSD curve of the N-RTLS algorithm reached at the steady state, the MSD curve of the proposed VFF-RTLS algorithm decreased continuously (Fig. 1).

An abrupt change of the system is introduced at iterations 5000. In this case, N-RTLS diverges for a while. However, our VFF-RTLS provides good tracking capability because the variable forgetting factor allows our algorithm to respond sensitively to changes in the system (Fig. 2).

From this result, our VFF-RTLS algorithm estimated the FIR of the unknown system h more precisely than does the N-RTLS algorithm which uses a fixed forgetting factor for both of stationary and non-stationary environments.

3. Conclusion

This study proposed the VFF-RTLS algorithm which uses a variable forgetting factor to solve the TLS problem for adaptive FIR filtering. Our method of varying the forgetting factor was described and compared with the existing N-RTLS algorithm. In computer simulation, the proposed VFF-RTLS algorithm achieved better estimation of the FIR of the unknown system than did the N-RTLS algorithm.

The adaptive filtering algorithm has been used in numerous adaptive signal processing applications including echo cancelation, and system identification. Because precise estimation of the FIR of an unknown system is important, the proposed VFF-RTLS algorithm will be useful in adaptive signal processing applications.

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5. References

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