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New Analytical Solutions for Nonlinear Heat Conduction Equation

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Abstract. In this paper, we derive exact traveling wave solutions of nonlinear heat conduction equation by a proposed Bernoulli sub-ODE method and the known (G'/G) expansion method.

Keywords: Bernoulli sub-ODE method, (G'/G) Expansion Method, traveling wave solutions, exact solution, evolution equation, nonlinear heat conduction equation

1. Introduction

Nonlinear evolution equations (NLEEs) have been the subject of study in various branches of mathematical-physical sciences such as physics, biology, chemistry, etc. The powerful and efficient methods to find analytic solutions and numerical solutions of nonlinear equations have drawn a lot of interest by many authors. Many efficient methods have been presented so far.

During the past four decades or so searching for explicit solutions of nonlinear evolution equations by using various different methods have been the main goal for many researchers, and many powerful methods for constructing exact solutions of nonlinear evolution equations have been established and developed such as are the homogeneous balance method [1,2], the hyperbolic tangent expansion method [3,4], the trial function method [5], the tanh-method [6-8], the nonlinear transform method [9], the inverse scattering transform [10], the Backlund transform [11,12], the Hirotas bilinear method [13,14], the generalized Riccati equation [15,16], the theta function method [17-19], the sine-Ccosine method [20], the Jacobi elliptic function expansion [21,22], the complex hyperbolic function method [23-25], and so on.

In this paper, we proposed a Bernoulli sub-ODE method to construct exact traveling wave solutions for NLEES.

The rest of the paper is organized as follows. In Section 2, we describe the Bernoulli sub-ODE method for finding traveling wave solutions of nonlinear evolution equations, and give the main steps of the method. In the subsequent sections, we will apply the Bernoulli Sub-ODE method and the known (G'/G) expansion method to find exact traveling wave solutions of the nonlinear heat conduction equation. In the last Section, some conclusions are presented.

2. Description of the Bernoulli Sub-ODE method

In this section we present the solutions of the following ODE:

$$G'+\lambda G=\mu G^2, \tag{2.1}$$

where $\lambda\neq 0, G=G(\xi)$

When $\mu\neq 0$, Eq. (2.1) is the type of Bernoulli equation, and we can obtain the solution as

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$$G=\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}},\tag{2.2}$$

where d is an arbitrary constant.

Suppose that a nonlinear equation, say in two or three independent variables x, y and t , is given by

$$P(u,u_{t,}u_x,u_y,u_{tt},u_{xt},u_{yt},u_{xx},u_{yy}......)=0\tag{2.3}$$

where u = u(x, y, t) is an unknown function, P is a polynomial in u = u(x, y, t) and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. By using the solutions of Eq. (2.1), we can construct a serials of exact solutions of nonlinear equations:.

*Step 1.*We suppose that

$$u(x,y,t)=u(\xi),\xi=\xi(x,y,t)\tag{2.4}$$

the traveling wave variable (2.4) permits us reducing Eq. (2.3) to an ODE for $u=u(\xi)$

$$P(u,u',u'',......)=0\tag{2.5}$$

Step 2. Suppose that the solution of (2.5) can be expressed by a polynomial in G as follows:

$$u(\xi)=\alpha_m G^m+\alpha_{m-1}G^{m-1}+......\tag{2.6}$$

where $G=G(\xi)$ satisfies Eq. (2.1), and $\alpha_m,\alpha_{m-1}...$ are constants to be determined later, $\alpha_m\neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (2.5).

Step 3. Substituting (2.6) into (2.5) and using (2.1), collecting all terms with the same order of G together, the left-hand side of Eq. (2.5) is converted into another polynomial in G . Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_m,\alpha_{m-1},...\lambda,\mu$.

Step 4. Solving the algebraic equations system in Step 3, and by using the solutions of Eq. (2.1), we can construct the traveling wave solutions of the nonlinear evolution equation (2.5).

In the subsequent sections we will illustrate the proposed method in detail by applying it to nonlinear heat conduction equation.

3. Application Of the Bernoulli Sub-ODE Method For nonlinear heat conduction Equation

In this section, we will consider the following nonline-ar heat conduction equation:

$$u_t-\alpha(u^n)_{xx}-u+u^n=0\tag{3.1}$$

In order to obtain the traveling wave solutions of Eq. (3.1), we suppose that

$$u(x,t)=u(\xi),\xi=kx+\omega t\tag{3.2}$$

where k,ω are constants that to be determined later. By using (3.2), (3.1) is converted into an ODE

$$\omega u'-ak^2(u^n)''-u+u^n=0\tag{3.3}$$

Suppose that the solution of (3.3) can be expressed by a polynomial in G as follows:

$$u(\xi)=\sum_{i=0}^{m}a_iG^i\tag{3.4}$$

where a_i are constants. Balancing the order of u' and u^n in Eq. (3.3), we have

$m+1=mn+2\Rightarrow m=-\frac{1}{n-1}$. If we make a variable $u=v^{-\frac{1}{n-1}}$, then (3.3) is converted into

$$\frac{\omega(n-1)}{3}(v^3)'+(n-1)^2v^2(v-1)+ak^2n(2n-1)(v')^2-ak^2n(n-1)vv''=0\tag{3.5}$$

Suppose that the solution of (3.5) can be expressed by a polynomial in G as follows:

$$v(\xi)=\sum_{i=0}^{l}b_iG^i \tag{3.6}$$

where b_i are constants, $G=G(\xi)$ satisfies Eq. (2.1). Balancing the order of $(v^3)'$ and vv'' in Eq. (3.5), we have $3l+1=l+l+2\Rightarrow l=1$ .So Eq. (3.6) can be rewritten as

$$v(\xi)=b_1G+b_0,b_1\neq 0 \tag{3.7}$$

where b_1,b_0 are constants to be determined later.

Substituting (3.7) into (3.5) and collecting all the terms with the same power of G together and equating each coefficient to zero, yields a set of simultaneous algebraic equations. Solving the algebraic equations above, yields:

Case 1:

$$b_1=\frac{q\mu}{p\lambda},b_0=0,\omega=-\frac{p}{2\lambda},k=\frac{1}{2}\sqrt{\frac{q}{\lambda}} \tag{3.8}$$

Substituting (3.8) into (3.7), we have

$$v_1(\xi)=\frac{q\mu}{p\lambda}G\text{, }\xi=\frac{1}{2}\sqrt{\frac{q}{\lambda}}x-\frac{p}{2\lambda}t \tag{3.9}$$

Substituting the general solutions of (2.1) into (3.9), we have

:

$$v_1(\xi)=\frac{q\mu}{p\lambda}(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}})$$

Then

$$u_1(\xi)=(v_1(\xi))=[\frac{q\mu}{p\lambda}(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}})]^{-\frac{1}{n-1}}$$

where $\xi=\pm\frac{1}{2}\sqrt{\frac{q}{\lambda}}x-\frac{p}{2\lambda}t\cdot C_1$ and C_2 are two arbitrary constants.

Case 2:

$$b_1=\frac{q\mu}{p\lambda},b_0=0,\omega=-\frac{p}{2\lambda},k=-\frac{1}{2}\sqrt{\frac{q}{\lambda}} \tag{3.10}$$

Substituting (3.8) into (3.7), we have

$$v_2(\xi)=\frac{q\mu}{p\lambda}G\text{, }\xi=-\frac{1}{2}\sqrt{\frac{q}{\lambda}}x-\frac{p}{2\lambda}t \tag{3.11}$$

Substituting the general solutions of (2.1) into (3.11), we have

$$v_2(\xi)=\frac{q\mu}{p\lambda}(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}})$$

Then

$$u_2(\xi)=(v_2(\xi))=[\frac{q\mu}{p\lambda}(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}})]^{-\frac{1}{n-1}}$$

where $\xi=-\frac{1}{2}\sqrt{\frac{q}{\lambda}}x-\frac{p}{2\lambda}t\cdot C_1$ and C_2 are two arbitrary constants.

Case 3:

$$b_1=-\frac{q\mu}{p\lambda},b_0=\frac{q}{p},\omega=-\frac{p}{2\lambda},k=\frac{1}{2}\sqrt{\frac{q}{\lambda}} \tag{3.12}$$

Substituting (3.8) into (3.7), we have

$$v_3(\xi)=-\frac{q\mu}{p\lambda}G+\frac{q}{p},\ \xi=\frac{1}{2}\sqrt{\frac{q}{\lambda}}x-\frac{p}{2\lambda}t \tag{3.13}$$

Substituting the general solutions of (2.1) into (3.13), we have

$$v_3(\xi)=-\frac{q\mu}{p\lambda}(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}})+\frac{q}{p}$$

Then

$$u_3(\xi)=(v_3(\xi))=[-\frac{q\mu}{p\lambda}(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}})+\frac{q}{p}]^{-\frac{1}{n-1}}$$

where $\xi=\frac{1}{2}\sqrt{\frac{q}{\lambda}}x-\frac{p}{2\lambda}t\cdot C_1$ and C_2 are two arbitrary constants.

4. Application Of (G'/G) expansion Method For nonlinear heat conduction Equation

In this section, we apply the (G'/G) expansion method to obtain the traveling wave solutions of nonlinear heat conduction equation (3.1).

Suppose that the solution of (3.5) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$v(\xi)=\sum_{i=0}^{l}b_i(\tfrac{G'}{G})^i \tag{4.1}$$

where b_i are constants, $G=G(\xi)$ satisfies the second order LODE in the form:

$$G''+\lambda G'+\mu G=0 \tag{4.2}$$

where λ and μ are constants. Balancing the order of $(v^3)'$ and vv'' in Eq. (3.5), we have $3l+1=l+l+2\Rightarrow l=1$ .So Eq. (4.6) can be rewritten as

$$v(\xi)=b_1(\tfrac{G'}{G})+b_0,b_1\neq 0 \tag{4.3}$$

where b_1,b_0 are constants to be determined later.

Substituting (4.3) into (3.5) and collecting all the terms with the same power of $(\frac{G'}{G})$ together and equating each coefficient to zero, yields a set of simultaneous algebraic equations. Solving the algebraic equations above, yields:

Case 1: when $\lambda^2-4\mu>0$

$$b_1 = \pm\sqrt{\frac{1}{\lambda^2-4\mu}}, b_0 = \pm\frac{1}{2}\sqrt{\frac{1}{\lambda^2-4\mu}}+\frac{1}{2},\ k = \pm\frac{n-1}{n}\sqrt{\frac{1}{a\lambda^2-4a\mu}}, \omega = \pm\frac{(n-1)\sqrt{\lambda^2-4\mu}}{n(\lambda^2+4\mu)} \tag{4.4}$$

Substituting (4.4) into (4.3), we have

$$v(\xi) = \pm\sqrt{\frac{1}{\lambda^2-4\mu}}(\frac{G'}{G})\pm\frac{1}{2}\sqrt{\frac{1}{\lambda^2-4\mu}}+\frac{1}{2},\ \xi = \pm\frac{n-1}{n}\sqrt{\frac{1}{a\lambda^2-4a\mu}}x\pm\frac{(n-1)\sqrt{\lambda^2-4\mu}}{n(\lambda^2+4\mu)}t \tag{4.5}$$

Substituting the general solutions of (4.2) into (4.5), we have:

$$v_1(\xi) = \mp\frac{\lambda}{2}\sqrt{\frac{1}{\lambda^2-4\mu}}\pm\frac{1}{2}(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})\pm\frac{1}{2}\sqrt{\frac{1}{\lambda^2-4\mu}}+\frac{1}{2}$$

Then

$$u_1(\xi) = (v_1(\xi))^{-\frac{1}{n-1}}$$

where $\xi = \pm\frac{n-1}{n}\sqrt{\frac{1}{a\lambda^2-4a\mu}}x\pm\frac{(n-1)\sqrt{\lambda^2-4\mu}}{n(\lambda^2+4\mu)}t$, and C_1 and C_2 are two arbitrary constants.

Case 2: when $\lambda^2-4\mu<0$

$$b_1 = \pm\sqrt{\frac{1}{4\mu-\lambda^2}}i, b_0 = \pm\frac{1}{2}\sqrt{\frac{1}{4\mu-\lambda^2}}i+\frac{1}{2},\ k = \pm\frac{n-1}{n}\sqrt{\frac{1}{4a\mu-a\lambda^2}}i, \omega = \pm\frac{(n-1)\sqrt{4\mu-\lambda^2}i}{n(\lambda^2+4\mu)} \tag{4.6}$$

Substituting (4.6) into (4.3), we have

$$v(\xi) = \pm\sqrt{\frac{1}{4\mu-\lambda^2}}i(\frac{G'}{G})\pm\frac{1}{2}\sqrt{\frac{1}{4\mu-\lambda^2}}i+\frac{1}{2},\ \xi = \pm\frac{n-1}{n}\sqrt{\frac{1}{4a\mu-a\lambda^2}}ix\pm\frac{(n-1)\sqrt{4\mu-\lambda^2}i}{n(\lambda^2+4\mu)}t \tag{4.7}$$

Substituting the general solutions of (4.2) into (4.7), we have:

$$v_2(\xi) = \mp\frac{\lambda}{2}\sqrt{\frac{1}{4\mu-\lambda^2}}i\pm\frac{1}{2}i(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})\pm\frac{1}{2}\sqrt{\frac{1}{4\mu-\lambda^2}}i+\frac{1}{2}$$

Then

$$u_2(\xi) = (v_2(\xi))^{-\frac{1}{n-1}},$$

where $\xi = \pm\frac{n-1}{n}\sqrt{\frac{1}{4a\mu-a\lambda^2}}ix\pm\frac{(n-1)\sqrt{4\mu-\lambda^2}i}{n(\lambda^2+4\mu)}t$, and C_1 and C_2 are two arbitrary constants.

Remark: As one can see from Section III and Section IV, the traveling wave solutions obtained by the Bernoulli Sub-ODE method are different from those by the known (G'/G) expansion method.

5. Conclusions

We have seen that some new traveling wave solutions of nonlinear heat conduction equation are successfully found by using the Bernoulli sub-ODE method. The main points of the method are that assuming the solution of the ODE reduced by using the traveling wave variable as well as integrating can be expressed by an m-th degree polynomial in G, where $G = G(\xi)$ is the general solutions of a Bernoulli sub-ODE equation. The positive integer m can be determined by the general homogeneous balance method, and the coefficients of the polynomial can be obtained by solving a set of simultaneous algebraic equations. The Bernoulli Sub-ODE method can be applied to many other nonlinear problems.

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