2011 International Conference on Computer and Automation Engineering (ICCAE 2011)
IPCSIT vol. 44 (2012) © (2012) IACSIT Press, Singapore
DOI: 10.7763/IPCSIT.2012.V44.12

Viscosity Solutions with Asymptotic Behavior of Hessian Quotient Equations

Limei Dai +

School of Mathematics and Information Science Weifang University
Weifang 261061, The People's Republic of China

Abstract. In this paper, we use the Perron method to prove the existence of viscosity solutions with asymptotic behavior at infinity to Hessian quotient equations.

Keywords: viscosity solutions, asymptotic behaviour, Hessian quotient equations

1. Introduction

In this paper, we study the Hessian quotient equation

$$S_{k,l}(D^2u) = \frac{S_k(D^2u)}{S_l(D^2u)} = 1 \quad \text{in } \mathrm{R}^n \setminus \partial\Omega . \tag{1}$$

Here $\Omega \subset\subset \mathrm{R}^n$ is any strictly convex bounded domain, $0 \le l < k \le n$, D^2u denotes the Hessian matrix of the function u, and $S_j(D^2u)$ is defined to be the jth elementary symmetric function of the eigenvalues $\lambda = (\lambda_1, \cdots, \lambda_n)$ of D^2u, that is,

$$S_j(D^2u) = \sigma_j(\lambda(D^2u)) = \sum_{1\le i_1<\cdots<i_j\le n} \lambda_{i_1}\cdots\lambda_{i_j}, \; j = 1,2,\cdots,n.$$

When $l = 0$, we denote $S_0(D^2u) \equiv 1$.

Hessian quotient equation (1) is an important class of fully nonlinear elliptic equation which is closely related to geometric problem. Some well-known equations can be regarded as its special cases. When $l = 0$, it is a k-Hessian equation. In particular, it is a Poisson equation if $k = 1$, while it is a Monge-Ampère equation if $k = n$. When $k = n = 3, l = 1$, that is, $\det(D^2u) = \Delta u$, (1) arises from special Lagrangian geometry([1]): if u is a solution of (1), the graph of Du over R^3 in C^3 is a special Lagrangian submanifold in C^3, that is, its mean curvature vanishes everywhere and the complex structure on C^3 sends the tangent space of the graph to the normal space at every point. Therefore (1) has drawn much attention, see [2-5].

The entire solutions of PDE have been studied by many authors, see [6,7]. In [CL], Caffarelli and Li have investigated the existence of solutions with asymptotic behavior to Monge-Ampère equations in exterior domain. Dai in [D] has proved the existence of viscosity solutions with asymptotic behavior to Hessian equations in exterior domain. In this paper, we study the viscosity solutions with asymptotic behavior at infinity to Hessian quotient equation (1) in $\mathrm{R}^n \setminus \partial\Omega$.

To work in the realm of elliptic equations, we have to restrict the class of functions. Let

$$\mathbf{\Gamma}_k = \left\{\lambda \in \mathrm{R}^n \middle| \sigma_j(\lambda) > 0, j = 1,2,\cdots,k\right\}.$$

+ Corresponding author.
E-mail address: limeidai@yahoo.com.cn

A function $u \in C^2(\mathrm{R}^n \setminus \partial\Omega)$ is called k-convex (uniformly k-convex) if $\lambda \in \overline{\Gamma}_k(\Gamma_k)$, where $\lambda = \lambda(D^2u) = (\lambda_1, \cdots, \lambda_n)$ is the eigenvalues of the Hessian matrix D^2u.

From [2] and [3], we know that (1) is elliptic and

$$(S_{k,l}(D^2u))^{\frac{1}{k-l}} = \left(\frac{S_k(D^2u)}{S_l(D^2u)}\right)^{\frac{1}{k-l}}$$

is a concave function of the second derivatives of u if u is uniformly k-convex. It is natural for the solutions of (1) to be considered in the class of uniformly k-convex functions.

An extensive study of viscosity solutions of second order partial differential equations can be found in [8] and [9].

For the reader's convenience, we recall the definition of viscosity solutions to Hessian quotient equations.

A function $u \in C^0(\mathrm{R}^n \setminus \partial\Omega)$ is called a viscosity subsolution of (1), if for any $y \in \mathrm{R}^n \setminus \partial\Omega$, $\xi \in C^2(\mathrm{R}^n \setminus \partial\Omega)$ satisfying

$$u(x) \le \xi(x), x \in \mathrm{R}^n \setminus \partial\Omega, \text{ and } u(y) = \xi(y)$$

we have

$$S_{k,l}(D^2\xi(y)) \ge 1.$$

A function $u \in C^0(\mathrm{R}^n \setminus \partial\Omega)$ is called a viscosity supersolution of (1), if for any $y \in \mathrm{R}^n \setminus \partial\Omega$, any k-convex function $\xi \in C^2(\mathrm{R}^n \setminus \partial\Omega)$ satisfying

$$u(x) \ge \xi(x), x \in \mathrm{R}^n \setminus \partial\Omega, \text{ and } u(y) = \xi(y)$$

we have

$$S_{k,l}(D^2\xi(y)) \le 1.$$

A function $u \in C^0(\mathrm{R}^n \setminus \partial\Omega)$ is called a viscosity solution of (1), if u is both a viscosity subsolution and a viscosity supersolution of (1).

A function $u \in C^0(\mathrm{R}^n \setminus \partial\Omega)$ is called k-convex if in the viscosity sense $\sigma_j(\lambda(D^2u)) \ge 0$ in $R^n \setminus \partial\Omega$, $j = 1, 2, \cdots, k$.

$u \in C^0(\mathrm{R}^n \setminus \partial\Omega)$ is k-convex if and only if u is C^0 subharmonic; u is n-convex if and only if u is convex.

2. Main Results

From Proposition 2.2 in [9], we know the supremum of a set of subsolutions is still a subsolution. Moreover, a comparison principle of viscosity solutions to Hessian quotient equations holds, see Theorem 3.3 in [8]. Then we can state the following existence and uniqueness results, see Proposition 2.3 in [9].

Lemma 1 Let B be a ball in R^n and $f \in C^0(\overline{B})$ be nonnegative. Suppose $\underline{u}, \overline{u} \in C^0(\overline{B})$ are respectively viscosity subsolution and supersolution of

$$S_{k,l}(D^2u) = f \quad \text{in} \quad B, \tag{2}$$

and satisfy $\underline{u}|_{\partial B} = \overline{u}|_{\partial B} = \varphi \in C^0(\partial B)$, then there exists a unique k-convex function $u \in C^0(\overline{B})$ satisfying (2) and $u = \varphi$ on ∂B.

Lemma 2 Let B be a ball in R^n and $f \in C^0(\overline{B})$ be nonnegative. Suppose $u \in C^0(\overline{B})$ satisfy in the viscosity sense $S_{k,l}(D^2\underline{u}) \ge f$ in B. Then the Dirichlet problem

$$S_{k,l}(D^2u) = f \quad \text{in} \quad B,$$
$$u = \underline{u} \quad \text{on} \quad \partial B$$

has a unique k-convex viscosity solution $u \in C^0(\overline{B})$.

Lemma 3 Let D be an open set in R^n and $f \in C^0(\mathrm{R}^n)$ be nonnegative. Assume k-convex functions $v \in C^0(\overline{D})$, $u \in C^0(\mathrm{R}^n)$ satisfy respectively

$$S_{k,l}(D^2v) \ge f(x), x \in D,$$
$$S_{k,l}(D^2u) \ge f(x), x \in \mathrm{R}^n.$$

Moreover,

$$u \le v, x \in \overline{D},$$
$$u = v, x \in \partial D.$$

Set

$$w(x) = \begin{cases} v(x), x \in D, \\ u(x), x \in \mathrm{R}^n \setminus D. \end{cases}$$

Then $w \in C^0(\mathrm{R}^n)$ satisfies in the viscosity sense

$$S_{k,l}(D^2 w) \ge f(x), x \in \mathrm{R}^n .$$

Lemma 4 Let $\Omega \subset R^n$ be a bounded strictly convex domain, $\partial\Omega \in C^2$, $v \in C^2(\overline{\Omega})$. Then there exists a constant c only dependent of $n, \Omega, \|v\|_{C^2(\Omega)}$ such that for any $\zeta \in \partial\Omega$ there exists $\bar{x}(\zeta) \in \mathrm{R}^n$ satisfying

$$|\bar{x}(\zeta)| \le c, w_\zeta(x) < v(x), x \in \overline{\Omega} \setminus \{\zeta\} ,$$

where

$$w_\zeta(x) = v(\zeta) + \frac{c_*}{2}\left(|x - \bar{x}(\zeta)|^2 - |\zeta - \bar{x}(\zeta)|^2\right), x \in \mathrm{R}^n ,$$

$$c_* = \left(C_n^l \Big/ C_n^k \right)^{\frac{1}{k-l}} .$$

Our main result is the following theorem.

Theorem 1 Let $k - l \ge 3$. For any $c \in \mathrm{R}$, there exists a constant $\beta_0 \in \mathrm{R}$ such that for any $\beta > \beta_0$ there exists a k - convex viscosity solution $u \in C^0(\mathrm{R}^n \setminus \partial\Omega)$ of (1) which satisfies

$$\limsup_{|x| \to \infty} \left(|x|^{k-l-2} \left| u(x) - \left(\frac{c_*}{2}|x|^2 + b \cdot x + c \right) \right| \right) < \infty \tag{4}$$

$$u(x) = -\beta, x \in \partial\Omega ,$$

where $c_* = \left(C_n^l \Big/ C_n^k \right)^{\frac{1}{k-l}}$ and Ω is any convex domain in R^n.

Proof. By subtracting a linear function from u, we only need to prove the theorem for $b = 0$. We divide the proof into three steps.

In the first step, we construct a viscosity subsolution of (1).

Let Ω be any bounded strictly convex domain in R^n .Suppose $\Phi \in C^{3,\alpha}(\overline{\Omega})$ (see [3]) is a k -convex function satisfying

$$S_{k,l}(D^2\Phi) = c_0 > 1, x \in \Omega,$$
$$\Phi = 0, x \in \partial\Omega .$$

By the comparison principle, $\Phi \le 0$ in Ω. And by Lemma 4, for each $\zeta \in \partial\Omega$, there exists $\bar{x}(\zeta) \in \mathrm{R}^n$ such that

$$w_\zeta(x) < \Phi(x), x \in \overline{\Omega} \setminus \{\zeta\} ,$$

where

$$w_\zeta(x) = \frac{c_*}{2}\left(|x - \bar{x}(\zeta)|^2 - |\zeta - \bar{x}(\zeta)|^2\right), x \in \mathrm{R}^n .$$

and $\sup_{\zeta \in \partial\Omega} |\bar{x}(\zeta)| < \infty$. Therefore

$$w_\zeta(\zeta) = 0, w_\zeta(x) \le \Phi(x) \le 0, x \in \overline{\Omega} .$$
$$S_{k,l}(D^2 w_\zeta(x)) = 1, x \in \mathrm{R}^n.$$

Thus

$$w(x) = \sup_{\zeta \in \partial\Omega} w_\zeta(x), x \in \mathrm{R}^n$$

satisfies

$$w(x) \le \Phi(x), x \in \Omega , \tag{5}$$

and from Proposition in [9],

$$S_{k,l}(D^2 w) \ge 1, x \in \mathrm{R}^n .$$

Define

$$V(x) = \begin{cases} \Phi(x), x \in \Omega, \\ w(x), x \in \mathrm{R}^n \setminus \Omega. \end{cases}$$

Then $V \in C^0(\mathrm{R}^n)$. By (5) and Lemma 3, V satisfies in the viscosity sense

$$F(D^2 V) \ge 1, x \in \mathrm{R}^n .$$

Fix some $R_1 > 0$ such that

$$\Omega \subset\subset B_{R_1} ,$$

and let

$$R_2 = 2R_1\sqrt{c_*} .$$

For $a > 1$, define

$$w_a(x) = \inf_{B_{R_1}} V + \int_{2R_2}^{|\sqrt{c_*}x|} (s^{k-l} + a)^{\frac{1}{k-l}} ds, x \in \mathrm{R}^n.$$

A direct calculation gives for $|x| > 0$,

$$D_{ij} w_a = \left(|y|^{k-l} + a\right)^{\frac{1}{k-l}-1}\left[\left(|y|^{k-l-1} + \frac{a}{|y|}\right)c_*\delta_{ij} - \frac{ac_*^2 x_i x_j}{|y|^3}\right],$$

where $y = \sqrt{c_*}x$. By rotating the coordinates, we may set $x = (r, 0, \cdots, 0)'$, therefore

$$D^2 w_a = \left(R^{k-l} + a\right)^{\frac{1}{k-l}-1}\begin{pmatrix} R^{k-l-1}c_* & 0 & \cdots & 0 \\ 0 & \frac{R^{k-l}+a}{R}c_* & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{R^{k-l}+a}{R}c_* \end{pmatrix},$$

where $R = |y|$. So $\lambda(D^2 w_a) \in \Gamma_k$. Let $R_0 = R^{k-l} + a$, then

$$\begin{aligned} S_{k,l}(D^2 w_a) &= \frac{S_k(D^2 w_a)}{S_l(D^2 w_a)} \\ &= \frac{R_0^{\frac{k}{k-l}-k}\left[C_{n-1}^k\left(\frac{R_0 c_*}{R}\right)^k + R^{k-l-1}c_* C_{n-1}^{k-1}\left(\frac{R_0 c_*}{R}\right)^{k-1}\right]}{\mathrm{R}_0^{\frac{1}{\mathrm{k}-1}-l}\left[C_{n-1}^l\left(\frac{R_0 c_*}{R}\right)^l + R^{k-l-1}c_* C_{n-1}^{l-1}\left(\frac{R_0 c_*}{R}\right)^{l-1}\right]} \\ &= R_0 c_*^{k-l} R^{l-k} \frac{C_n^k R^{k-l} + aC_{n-1}^k}{C_n^l R^{k-l} + aC_{n-1}^l} \\ &\ge R_0 c_*^{k-l} R^{l-k} \frac{C_n^k R^{k-l}}{C_n^l R^{k-l} + aC_n^l} \\ &= c_*^{k-l}\frac{C_n^k}{C_n^l} = 1. \end{aligned}$$

By the definition of R_2,

$$\left|\sqrt{c_*}x\right| \le \frac{R_2}{2}, |x| \le R_1 .$$

Consequently

$$\begin{aligned} w_a(x) &\le \inf_{B_{R_1}} V + \int_{2R_2}^{\frac{R_2}{2}} \left(s^{k-l} + a\right)^{\frac{1}{k-l}} ds \\ &< \inf_{B_{R_1}} V \le V(x), |x| \le R_1 . \end{aligned} \tag{6}$$

Fix some $R_3 > 3R_2$ satisfying

$$R_3\sqrt{c_*} > 3R_2 .$$

We choose $a_1 > 1$ such that for $a \ge a_1$,

$$\begin{aligned} w_a(x) &> \inf_{B_{R_1}} V + \int_{2R_2}^{3R_2} \left(s^{k-l} + a\right)^{\frac{1}{k-l}} ds \\ &\ge V(x), |x| = R_3 . \end{aligned}$$

Then by (6),

$$R_3 \ge R_1 .$$

By the definition of w_a,

$$\begin{aligned} w_a(x) &= \inf_{B_{R_1}} V + \int_{2R_2}^{|\sqrt{c_*}x|} s\left(\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right)ds + \int_{2R_2}^{|\sqrt{c_*}x|} sds \\ &= \inf_{B_{R_1}} V + \int_{2R_2}^{|\sqrt{c_*}x|} s\left(\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right)ds + \frac{c_*}{2}|x|^2 - 2R_2^2 \\ &= \frac{c_*}{2}|x|^2 + c + \inf_{B_{R_1}} V + \int_{2R_2}^{\infty} s\left(\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right)ds \\ &\quad - c - 2R_2^2 - \int_{|\sqrt{c_*}x|}^{\infty} s\left(\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right)ds, x \in R^n . \end{aligned}$$

Let

$$\mu(a) = \inf_{B_{R_1}} V + \int_{2R_2}^{\infty} s\left(\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right)ds - c - 2R_2^2 .$$

Then $\mu(a)$ is continuous and monotonic increasing for a and when $a \to \infty$, $\mu(a) \to \infty$. And,

$$w_a(x) - \mu(a) \le \frac{c_*}{2}|x|^2 + c, a \ge a_1, x \in \mathrm{R}^n .$$

Moreover, when $|x| \to \infty$,

$$w_a(x) = \frac{c_*}{2}|x|^2 + c + \mu(a) + O\left(|x|^{2-k+l}\right) \tag{7}$$

For $a \ge a_1$, set $\beta_0 = \mu(a)$ and define for any $\beta > \beta_0$,

$$\underline{u}_a(x) = \begin{cases} \max\{V(x), w_a(x)\} - \beta, |x| \le R_3, \\ w_a(x) - \beta, |x| \ge R_3 . \end{cases}$$

Then

$$\underline{u}_a(x) = -\beta, x \in \partial\Omega,$$

and by (7), when $|x| \to \infty$,

$$\underline{u}_a(x) = \frac{c_*}{2}|x|^2 + c + O\left(|x|^{2-k+l}\right) \tag{8}$$

Choose $a_2 \ge a_1$ sufficiently large such that for $a \ge a_2$,

$$\begin{aligned} V(x) - \beta &\le V(x) - \beta_0 \\ &= V(x) - \inf_{B_{R_1}} V + c + 2R_2^2 - \int_{2R_2}^{\infty} s\left(\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right)ds \\ &\le c \le \frac{c_*}{2}|x|^2 + c, |x| \le R_3 . \end{aligned}$$

Therefore

$$\underline{u}_a(x) \le \frac{c_*}{2}|x|^2 + c, a \ge a_2, x \in \mathrm{R}^n .$$

By Lemma 3, $\underline{u}_a(x) \in C^0(\mathrm{R}^n)$ satisfies in the viscosity sense

$$S_{k,l}(D^2\underline{u}_a) \ge 1, x \in \mathrm{R}^n .$$

In the second step, we define the Perron solution of (1).

For $a \ge a_2$, let S_a denote the set of functions $v \in C^0(\mathrm{R}^n)$ satisfying

$$S_{k,l}(D^2 v) \ge 1, x \in \mathrm{R}^n,$$
$$v(x) = -\beta, x \in \partial\Omega,$$
$$v(x) \le \frac{c_*}{2}|x|^2 + c, x \in \mathrm{R}^n.$$

Clearly, $\underline{u}_a \in S_a$. Hence $S_a \ne \phi$ .Define in R^n,

$$u_a(x) = \sup\{v(x) | v \in S_a\}.$$

By the definition of u_a, u_a is a viscosity subsolution of (1) and satisfies

$$u_a(x) = -\beta, x \in \partial\Omega,$$
$$u_a(x) \le \frac{c_*}{2}|x|^2 + c, x \in \mathrm{R}^n .$$

In the following, we prove u_a is a viscosity supersolution of (1).

In the third step, we prove u_a is a viscosity solution of (1) satisfying (4).

For any $x_0 \in \mathrm{R}^n \setminus \partial\Omega$, $\varepsilon > 0$, choose a ball $B = B_\varepsilon(x_0) \subset \mathrm{R}^n \setminus \partial\Omega$. By Lemma 2, the Dirichlet problem

$$S_{k,l}(D^2\tilde{u}) = 1, x \in B,$$
$$\tilde{u} = u_a, x \in \partial B$$

has a viscosity solution $\tilde{u} \in C^0(\overline{B})$. From the comparison principle, $u_a \le \tilde{u}, x \in B$. Define

$$\psi(x) = \begin{cases} \tilde{u}(x), x \in B, \\ u_a(x), x \in \mathrm{R}^n \setminus (\partial\Omega \cup B). \end{cases}$$

By Lemma 3,

$$S_{k,l}(D^2\psi(x)) \ge 1, x \in \mathrm{R}^n \setminus \partial\Omega .$$

Because

$$S_{k,l}(D^2\tilde{u}) = 1 = S_{k,l}(D^2 g), x \in B,$$
$$\tilde{u} = u_a \le g, x \in \partial B .$$

where $g(x) = \frac{c_*}{2}|x|^2 + c$, from the comparison principle,

$$\tilde{u} \le g, x \in B.$$

Hence $\psi \in S_a$. By the definition of u_a, $u_a \ge \psi$ in R^n. Consequently $\tilde{u} \le u_a$ in B. As a result,

$$\tilde{u} = u_a, x \in B.$$

Because x_0 is arbitrary, we know u_a is a viscosity supersolution of (1).

By the definition of u_a,

$$\underline{u}_a \le u_a \le g, x \in \mathrm{R}^n ,$$

so from (8), u_a satisfies (4). Theorem 1 is proved

3. Acknowledgements

The research was supported by Shandong Province Young and Middle-Aged Scientists Research Awards Fund(BS2011SF025), Shandong Province Science and Technology Development Project(2011YD16002), Shandong Province Natural Science Foundation(ZR2011AL008).

4. References

[1] R. Harvey and H. B. Lawson. Calibrated geometries, *Acta Math*. 1982, **148**: 47-157.

[2] L. Caffarelli , L. Nirenberg L and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic quations.III. Functions of the eigenvalues of the Hessian. *Acta Math*. 1985, **155**: 261-301.

[3] N. S. Trudinger, On the Dirichlet problem for Hessian equations. *Acta Math*. 1995, **175**: 151-164.

[4] J. G. Bao, J. Y. Chen, B. Guan, etal. Liouville property and regularity of a Hessian quotient equation. *Am. J. math*. 2003, **125**: 301-316.

[5] S. M. Liu and J. G. Bao. The local regularity for strong solutions of the Hessian quotient equation. *J. Math. Anal. Appl*. 2005, **303**: 462-476.

[6] L. Caffarelli L and Y. Y. Li. An extension to a theorem of Jörgens, Calabi, and Pogorelov. *Comm. Pure Appl. Math*. 2003, **56**: 549-58.

[7] L. M. Dai and J. G. Bao. Multi-valued solutions to fully nonlinear uniformly elliptic equations. *J. Math. Anal. Appl.* 2012, **389**: 314-321.

[8] M. G. Crandall , H. Ishii and P. L. Lions. User's guide to viscosity solutions of second order partial differential equations. *B. Am. Math .Soc*. 1992, **27**: 1-67.

[9] H. Ishii. On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDEs. *Comm. Pure Appl. Math.* 1989, **42**: 15-45.