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Satellite's Orbital Dynamic and Stable Regions near Equilibrium Points of Asteroid

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Abstract. In order to study the satellite's moving features and the size of stable reigns around the center equilibrium points of asteroid, 1:1 resonant orbits theory are applied in this paper. First the Hamilton function described by Delaunay variable is simplified to Schubart standard. Secondly the simplified mathematical model is used to discuss the form of orbital motion, vibration period of satellites which locate orbit the asteroid's equilibrium points and the relationship between each other. And then Asteroid Vesta is taken as a simulation example to verify the correctness of these conclusions above. Finally, exploiting the properties of energy integral of Schubart standard, the formula of deducing the radius of stable region around the center points is given, which is also applied to analyze the properties of orbital motions both inside and outside of the stable region. The research in this article not only can help us better understand the equilibrium points of asteroids and the dynamical environments in the vicinity of which but also can it supply theoretical basis in designing the orbit around the equilibrium points around the asteroid.

Keyword: asteroid, equilibrium point, orbital resonance, stable region;

1. Introduction

There exists four equilibrium points around an asteroid [1], these four points and the dynamical environment around them are quite similar to the Lagrange point in three-body system. D. J. Scheeres utilized ellipsoid integral function to find out the computational method of determining the positions of equilibrium points around the homogeneous triaxial ellipsoid and the stability conditions of equilibrium points, according to the stability conditions the equilibrium points can be classified into two classifications: Two unstable points which locates on both sides of semi-major axis of ellipsoid are named as Saddle Point; The other stable couple which locates on both sides of semi-minor axis are named as Center Point. Need to mention that the definitions of Saddle Point and Center Point here are different from those defined in Dynamic System, details can be checked in [1]. Saddle Point and Center Point in this paper obey the definition given by D. J. Scheeres. Due to trixial ellipsoid cannot decently reflect the properties of asteroid's gravity field and the computation of ellipsoid integral function is too complicated, the form of which is also obscure to help better understand the parameters of ellipsoid's shape and the relationship between the position and stability of center point, this method hasn't been widely applied, thus in order to solve the problems above, W.hu studied the related problem of equilibrium point in random second degree second order gravity field[2,3] and deduced the approximate calculation formula of positions of equilibrium points which are expressed by spherical harmonic coefficient, he also gave the stability criterion of equilibrium point. Although W. hu's work solved the problems existed in D. J. Scheeres's research, it's still not enough of only knowing the positions and stability of equilibrium points for an asteroid. Since "positioning the satellite at the asteroid's equilibrium point" cannot be realized in engineering, satellite's motion around the stable equilibrium point and the size of stable region become the principal problem which needs to figure out currently.

Changyin Zhao employed 1:1 orbit resonance theory in his research of stable region of triangular libration points (Lagrange points L_4, L_5 which are similar to the stable equilibrium points of asteroid)of planet in solar

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system[4,5]. Due to the dynamical environment around the stable center point is similar to the dynamical environment around Lagrange point L_4 L_5 of planet, 1:1 orbit resonance is adopted in this paper to study the related problems about satellite orbiting around the equilibrium point of asteroid. The problem can be concluded as two respects: First, the form of movement and moving characteristics of satellite whose initial position has a constant deviation from the center equilibrium point. Second, the problem about stable region around stable region around stable center equilibrium point, that is to say to control the initial deviation in what range can guarantee the satellite moving around the equilibrium point instead of moving far away.

2. The Position of Equilibrium Point and Its Stability

In the body-fixed frame of asteroid, the approximate computational formula of position of equilibrium point which utilizes normalized unit [6] can be expressed as

$$\begin{cases} x_{eq} \approx \pm\left(r_s + \dfrac{3C_{22} - 0.5C_{20}}{r_s}\right) \\ y_{eq} \approx \pm\left(r_s - \dfrac{3C_{22} + 0.5C_{20}}{r_s}\right) \end{cases} \tag{1}$$

Where $r_s = \omega_T^{-2/3}$, which is the synchronous orbit radius, C_{20} and C_{22} are the spherical harmonious coefficients of gravity field accord to the oblateness and ellipicity of center gravitational body. These four equilibrium points are saddle points $(\pm x_{eq}, 0)$ and center points $(0, \pm y_{eq})$. Equation (1) is the approximate computational formula of the stable position in second degree second order gravity field. Taking the results as initial value, by employing numerical method can the equilibrium position in random degree and order gravity filed be obtained, Fig.1 is the distribution map of the positions of equilibrium points in sixteenth order gravity field of asteroid Eros 433.

For the saddle point, it's quite unstable for the satellite moving in the vicinity of it. Only minor deviation can cause the satellite's orbit diverge in the form of index and finally the satellite will either escape from the asteroid or collide with the asteroid. Fig. 2 is the satellite's orbital condition, where the satellite's initial position is around the asteroid Eros 433's saddle point. The time of orbital recurrence is 0.947 day.

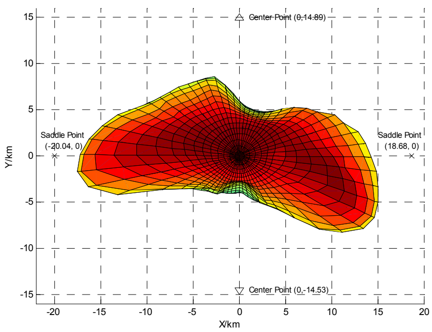

Fig. 1. Equilibrium points of Eros 433

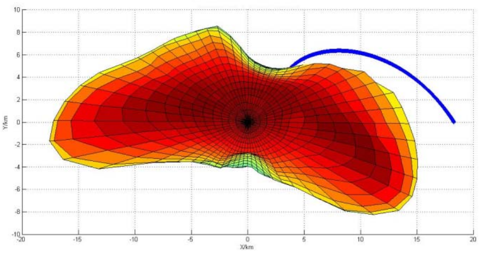

Fig. 2.Orbit with Initial Position in Saddle Point of Eros433

From Fig.2 we can see that the synchronous orbit didn't come into being, on the other hand, the satellite collided with the asteroid due to the perturbation within one day, which illustrates that the asteroid's saddle points are unstable, if the spherical harmonious coefficient of the asteroid satisfies equation (2)

$$r_s^2 + C_{20} - 162C_{22} > 0 \tag{2}$$

Table 1. The Stability of Equilibrium Point in Celestial Bodies

Center Body	Earth	Vesta	Eros433
$\mu(km^3/s^2)$	3.986×10^5	14.2525	4.462×10^{-4}
C_{20}	1.082×10^{-3}	-5.116×10^{-2}	-0.117
C_{22}	1.77×10^{-6}	8.31×10^{-2}	0.054
$\omega_T(rad/s)$	7.27×10^{-5}	3.29×10^{-4}	3.31×10^{-4}
$r_s(km)$	4.22×10^4	5.08×10^2	15.964
$r_s^2 + C_{20} - 162C_{22}$	$1.8\times10^9 > 0$	$1.9\times10^5 > 0$	$-2.0\times10^3 < 0$
Stability	Stable	Stable	Unstable

Then these two center points are stable [3], also it means that the satellite whose initial position locates at these two points will remain the position "at rest" relative to asteroid, even if there exists some deviation, the orbit of satellite cannot diverge. Table 1 lists the physical parameters like the mass, angular spin rate, oblateness and ellipicity of asteroid Vesta and Eros 433, also the stability criterion of center points in the form of equation (2) is given. Table 1 also lists the related data of center point of earth in comparison.

3. The Orbit Motion around Stable Center Point

The problems about positions of equilibrium points and stability of asteroid have been analyzed above. For the saddle points and unstable center points, the strongly instability of which can be utilized to design some escape orbit or capture orbit. For the stable center point, we can utilize its stability to arrange re-translator satellite or synchronous satellite which can be assigned a long-time observation mission of asteroid. In the following of this section the orbit motion around stable center point of asteroid is analyzed, due to the motion in the Z direction can resolve variables and is relatively easy to analyze[7], this paper only analyzes the satellite's orbit motion in the asteroid's equatorial plane.

Employing Delaunay variable, omitting the perturbation in higher than second-order, after the transformation, the short-period perturbation can be eliminated, then the Hamilton function can be obtained as [4,5]

$$\begin{aligned} H = & \left(\frac{1}{2L^2}+\omega_T L\right)-\frac{C_{20}}{4L^3G^3}\left(3\cos^2 i-1\right) \\ & +\frac{3C_{22}}{4L^6}(1+\cos i)^2\cos 2l \end{aligned} \tag{3}$$

L is a pair of conjugate variables, the definition of which is as follow

$$\begin{cases} L=\sqrt{a} \\ l=(\Omega+\omega+M)-\theta_{22}, \quad \theta_{22}=S_{22}+\omega_T(t-t_0) \end{cases} \tag{4}$$

Where t_0 is epoch time, S_{22} is the local sidereal time of x axis in body fixed frame of asteroid. And

$$\begin{cases} \beta=L, \quad \alpha=l-\dfrac{\pi}{2} \\ B(\beta)=\left(\dfrac{1}{2\beta^2}+\omega_T\beta\right)-\dfrac{C_{20}}{4\beta^6(1-e^2)^{3/2}}\left(3\cos^2 i-1\right) \\ \qquad -\dfrac{3C_{22}}{4\beta^6}(1+\cos i)^2 \\ A(\beta)=\dfrac{3C_{22}}{4\beta^6}(1+\cos i)^2 \end{cases} \tag{5}$$

Then Hamilton function can be written as standard Schubart model, which corresponds to a ideal resonance system with single DOF (β,α), the resonance variables are (β,α), α is the angle variable, β is the resonance variable which conjugates of α. The corresponding Hamilton function is

$$H(\beta,\alpha)=B(\beta)+2A(\beta)\sin^2\alpha \tag{6}$$

Where the second term of the right side of equation is the major resonance term, the relationship below is also tenable.

$$A(\beta)>0, \quad A(\beta)/|B(\beta)|=O(\varepsilon) \tag{7}$$

Where the small parameter ε indicates the magnitude of perturbation in Hamilton function, from the property of Hamilton function we know that the motion equation is

$$\begin{cases} \dot{\beta}=\dfrac{\partial H}{\partial\alpha}=P(\beta,\alpha)=2A(\beta)\sin 2\alpha \\ \dot{\alpha}=-\dfrac{\partial H}{\partial\beta}=Q(\beta,\alpha)=-B'(\beta)-2A'(\beta)\sin^2\alpha \end{cases} \tag{8}$$

Which has energy integral

$$H(\beta,\alpha)=C \tag{9}$$

Where C is the energy constant, differentiate the second equation in (8) by time

$$\ddot{\alpha}=(-B''-2A''\sin^2\alpha)\dot{\beta}-(2A'\sin 2\alpha)\dot{\alpha} \tag{10}$$

Introduce equation (8) into equation (10) can we get

$$\ddot{\alpha}=(-2AB''+2A'B')\sin 2\alpha+\left[-4AA''+(A')^2\right]\sin^2\alpha\sin 2\alpha \tag{11}$$

Around center equilibrium point $(0,\pm y_{eq})$

$$\begin{cases} B'\approx 0,\ B''=O(1) \\ A=O(\varepsilon),\ A'=O(\varepsilon),\ A''=O(\varepsilon) \end{cases} \tag{12}$$

Then introduce equation (10), expand sin2x in Taylor series at equilibrium point, we can obtain

$$\ddot{\alpha}=(-4AB'')\left[\alpha+O(\alpha^3)\right] \tag{13}$$

Around center equilibrium points $4AB''>0$, for $\omega_s^2=4AB''$ and omit the high-order term, the motion equation of original system can be simplified as simple harmonic motion equation

$$\ddot{\alpha}+\omega_s^2\alpha=0 \tag{14}$$

The solution of (14) is

$$\alpha=A_s\cos(\omega_s t+\alpha_0) \tag{15}$$

Where α_0 is the initial phase of α when t = 0, AZ is the amplitude, $\omega_s=\sqrt{4AB''}$ is vibration angular frequency. Equation (15) is the vibrating condition around center equilibrium points $(0,\pm y_{eq})$, which is the orbital resonance property in deep vibration region. From the relationship between period and angular frequency we can get

$$T_s=\frac{2\pi}{\omega_s}=\frac{2\pi}{\sqrt{4AB''}} \tag{16}$$

Table 2. The Stability of Equilibrium Point in Celestial Bodies

Swing	3.765	5.494	9.050	12.496	16.194
peroid	0.920	0.918	0.918	0.923	0.930
Swing	20.000	27.520	43.676	55.139	78.393
period	0.940	0.971	1.079	1.189	1.769

The analysis above indicates that when the spacecraft is moving in the vicinity of center equilibrium point, its mode of motion has the similar property with the mathematical pendulum as their periodicity, the resonance period is almost a constant which is nearly irrelevant to the amplitude, specific values can be calculated by equation (16). Table 2 gives the resonance amplitude of asteroid Vesta's synchronous orbit satellite and the corresponding period, form table 2 we know that in deep vibration region(amplitude is below 20 degree), the amplitude increases while period only changes a little. Thus when the amplitude keeps increasing till it reaches light vibration region, the increasing of amplitude will cause the period a vast change.

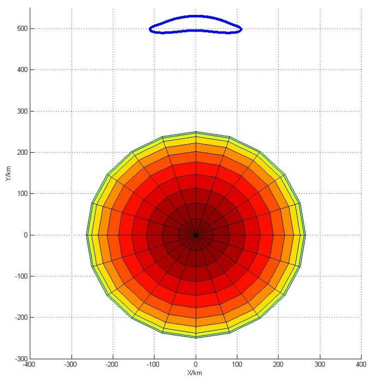

Fig. 3. Resonance Orbit in Deep Vibration Area of Vesta

Fig. 3 is the schematic illustration of Vesta's resonance orbit when amplitude is 12.5 degree, from Fig. 3 we know that in the body fixed frame of asteroid, satellite moves in the form of reciprocating vibration around center equilibrium point, the orbital shape is like a flat hoof. Fig. 4 is the resonance orbit when amplitude is 77.1 degree. It is seen from this that , when the amplitude is small, satellite can still be resonated steady around center equilibrium point in light vibration region, when the amplitude is oversized, the vibration becomes unstable, but the satellite still remains within the vicinity of equilibrium point.

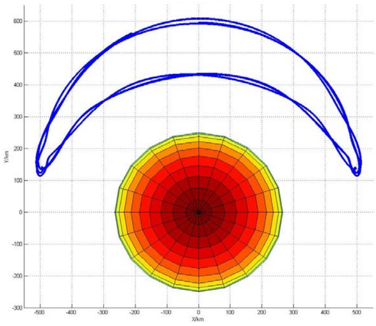

Fig. 4. Resonance Orbit in Light Vibration Area of Vesta

4. Stable Region of Orbit around the Equilibrium Point

Fig.4 indicates when the amplitude is oversized, the stability of resonance orbit in light vibration region becomes weakened. This section is going to deal with the problem of what extent the amplitude increases to will the resonance orbit becomes unstable. Equation (6) doesn't obviously contain *t*, thus β can be treated as the function of α, by differentiating equation (9) of α, we can get

$$\frac{d\beta}{d\alpha}=\frac{2A\sin 2\alpha}{-B'-2A'\sin^2\alpha}=\frac{P(\beta,\alpha)}{Q(\beta,\alpha)} \tag{17}$$

The point which equals zero in (17) is the stationary point of original function, if β acquires maximum then

$$\frac{2A\sin 2\alpha}{-B'-2A'\sin^2\alpha}=0$$

is for sure, but at the non-equilibrium point must have $B'+2A'\sin^2\alpha\neq 0$, hence the maximum of β must corresponds to $2A\sin 2\alpha=0$, that is to say $\alpha=0$. Form equation (6) and (9) we know on the boundary line of resonance region

$$C^*=B(\beta_s)+2A(\beta_s)=B(\beta^*) \tag{18}$$

Where β^* is the maximum of β on the boundary line, C^* is the corresponding energy constant, β_s is the value of β which corresponds to saddle point $(\pm x_{eq},0)$, due to the difference between β^* and β_s is very small, the $B(\beta^*)$ can be expanded as β_s 4, the expression is

$$B(\beta^*)=B(\beta_s)+B'(\beta_s)\Delta\beta+\frac{1}{2}B''(\beta_s)(\Delta\beta)^2+\cdots \tag{19}$$

Where $\Delta\beta=\beta^*-\beta_s$, is the stable resonance region, introduce equation (19) into (18)

$$\begin{aligned}&B(\beta_s)+2A(\beta_s)=B(\beta^*)\\&=B(\beta_s)+B'(\beta_s)\Delta\beta+\frac{1}{2}B''(\beta_s)(\Delta\beta)^2+o(\Delta\beta^3)\end{aligned} \tag{20}$$

We can see from the existence conditions of saddle point $Q(\beta_s,\alpha_s)=0$ that

$$B'(\beta_s)+2A(\beta_s)=0 \tag{21}$$

which can be introduced into (20) and then

$$\frac{1}{2}B''(\beta_s)(\Delta\beta)^2-2A'(\beta_s)\Delta\beta-2A(\beta_s)+o(\Delta\beta^3)=0 \tag{22}$$

Because of $B''=O(\varepsilon)$, $A=O(\varepsilon)$, $A'=O(\varepsilon)$, only $\Delta\beta=O\left(\varepsilon^{\frac{1}{2}}\right)$ can equation (22) be satisfied, omitting the high order terms we can obtain the reduced form

$$\frac{1}{2}B''(\beta_s)(\Delta\beta)^2 - 2A(\beta_s) = 0 \tag{23}$$

Therefore the approximate formula of stable resonance region can be expressed as

$$\Delta\beta = 2\sqrt{\frac{A}{B''}} \tag{24}$$

When calculating the resonance period T and stable resonance region $\Delta\beta$, B'' is very needed to calculate, thus the approximate calculation of B'' becomes important, differentiate the second equation in (5) twice can we get

$$B'' = 3L^{-4} + O(C_{20}) \tag{25}$$

Then we can finally obtain the resonance period and stable resonance region around the center equilibrium point by equation (5) (25) and (26), the final result of which is

$$\Delta\beta = \frac{1+\sin i}{2L}\sqrt{C_{22}} \tag{26}$$

Need to mention here is the definition of calculation result L from Delaunay variable is $L=\sqrt{a}$ if using standard unit, now we transform it into Caresian vector variable and introduce (1) into (26), then

$$\Delta y = \sqrt{\frac{C_{22}r_s}{r_s^2 - 0.5C_{20} + 3C_{22}}} \tag{27}$$

Equation (27) is the formula of stable resonance region in the equatorial plane of asteroid, Figure 5 is the conditions of resonance orbit of satellite locating on the edge of resonance region, from Fig. 5 we can see that after a resonance period the satellite is unable to return to its original position, which indicates the resonance orbit of which is already unstable, only some small perturbation or long-time orbital movement can cause the satellite moves far away from the center point.

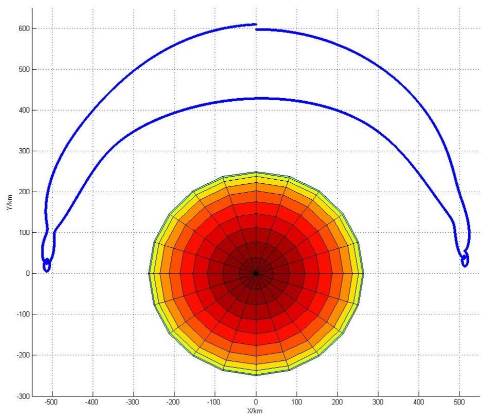

Fig. 5. Orbits in Stable Regions Edge of Vesta

5. Conclusion

This paper studied the properties of satellite's orbital motion around the stable center point of asteroid, which can be recognized as the inheriting and developing of what predecessors did. 1:1 orbit resonance theory is adopted to deduce the satellite's orbital motion around center equilibrium point (in the deep resonance region), which is in a resonance condition, the resonance state of angular motion is quite similar to the property of mathematical pendulum. The resonance period is almost a constant which is irrelevant to amplitude, the value of period can be calculated approximately by equation (16). The approximate width of stable resonance region can be obtained by equation (27). In this region the satellite's orbit is stable which means neither can satellite escape from asteroid nor collide with asteroid. If satellite exceeds the stable region, its orbit will become unstable, some small perturbation or long-time orbital motion may cause the satellite's escape or collision. The research of this paper can not only help us better understand the equilibrium point of satellite and the dynamical environment around it but also of great support in providing the theoretical basis when designing the orbit around equilibrium point of asteroid.

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