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Connected Dominating Set of Hypercubes and Star Graphs

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Abstract. In wired or wireless networks, routing efficiently among immobile or mobile devices is an important issue. A *connected dominating set* (*CDS*) brings benefits to network routing. The CDS can be served as a *virtual backbone* of a network, and it always adapted easily to new network topology. A virtual backbone is a set of vertices which can help with routing. Any vertex outside the virtual backbone can send messages or signals to another vertex through the virtual backbone. So the virtual backbone has great benefits to routing and management of networks. We may impose a virtual backbone to support short path routing, fault-tolerant routing, multi-casting, and radio broadcasting, etc. In this paper, we focus on constructing the *minimum CDS* (*MCDS*) of the *n*-dimensional hypercubes and *n*-dimensional star graphs.

Keywords: connected dominating set, virtual backbone, routing, hypercube, star graph.

1. Introduction

For the graph-theoretical terminology and notation, we follow [1]. $G = (V, E)$ is a simple graph if V is a finite set and E is a subset of $\{(u,v) \mid (u,v)$ is an unordered pair of $V\}$. We say that V is the vertex set and E is the edge set. Given a connected graph $G = (V, E)$ and a vertex set $R \subseteq V$. R is a *dominating set* (abbreviated as *DS*) if each vertex in G is either in R or has at least a neighbor in R; R is a *connected dominating set* (abbreviated as *CDS*) if R is a dominating set and the induced subgraph $G[R]$ is connected. A CDS with minimum cardinality is called a *minimum connected dominating set* (abbreviated as *MCDS*) [4,6,12]. Fig. 1 gives examples of the DS, CDS, and MCDS of a network G.

A connected dominating set (CDS) brings benefits to network routing. The CDS can be served as a *virtual backbone* of a network, and it always adapted easily to new network topology. A virtual backbone is a set of vertices which can help with routing. Any vertex outside the virtual backbone can send messages or signals to another vertex through the virtual backbone. So the virtual backbone has great benefits to routing and management of networks. We may impose a virtual backbone to support short path routing, fault-tolerant routing, multi-casting, and radio broadcasting, etc [7,18]. Furthermore, a virtual backbone of a wireless network may reduce communication overhead, increase bandwidth efficiency, and decrease energy consumption [13]. Therefore, the virtual backbone of a network with better network topologies benefits better performance of the network.

An ad-hoc network or a wireless sensor network is usually composed of a group of wireless vertices, the network can be considered as a *unit disk graph* [3], which is abbreviated as *UDG*. To obtain an MCDS in a UDG is an NP-hard problem [5]. Many literature references discussed the MCDS on UDG by *approximation algorithms* [4,6,7,8,16,17,18]. On some specific wired networks, there are also some related results. The MCDS of *meshes* was discussed in [10]; the CDS of *trapezoid graphs* and *generalized trapezoid graphs* was discussed in [17] and [11], respectively.

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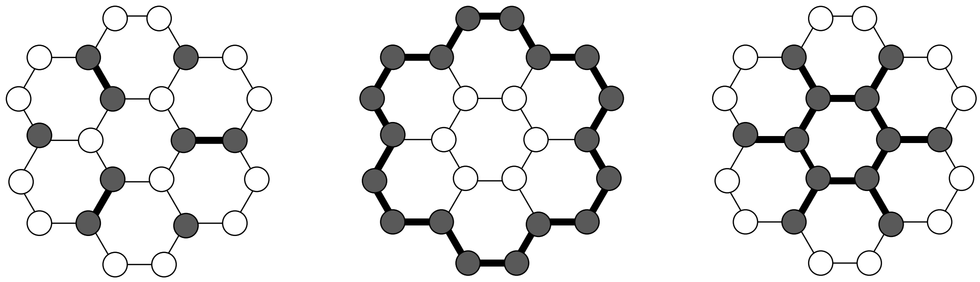

Fig. 1: (a) a DS of G; (b) a CDS of G; (c) an MCDS of G.

In this paper, we focus on constructing the MCDS of *n-dimensional hypercubes* and *n-dimensional star graphs*. In Section 2, two networks hypercubes and star graphs are introduced. In Section 3, for hypercubes and star graphs, we investigate in the upper bounds of cardinality of their MCDS. For the lower dimensional hypercubes and star graphs, we show the lower bounds of cardinality of their MCDS. Section 4 gives the concluding remarks.

2. Hypercubes and Star Graphs

The hypercube is a popular network because of its attractive properties, including regularity, symmetry, small diameter, strong connectivity, recursive construction, partitionability, and relatively low link complexity [14,15]. Let n be a positive integer. An n-dimensional hypercube, denoted by Q_n, is an n-regular graph with 2^n vertices. Each vertex v in Q_n can be distinctly labelled by an n-bit binary string, $v = v_n v_{n-1} \dots v_1$. For $1 \le i \le n$, we use v^i to denote the binary string $v_n v_{n-1} \dots \overline{v_i} \dots v_1$. The Q_n consists of all n-bit binary strings representing its vertices. Two vertices u and v are adjacent if and only if $v = u^i$ with some i. An n-dimensional hypercube Q_n can be constructed from two identical $(n-1)$-dimensional hypercubes, Q_{n-1}^0 and Q_{n-1}^1, where $V(Q_{n-1}^0) = \{v_n v_{n-1} \dots v_1 \mid v_n = 0\}$ and $V(Q_{n-1}^1) = \{v_n v_{n-1} \dots v_1 \mid v_n = 1\}$. The vertex set of Q_n is $V(Q_n) = V(Q_{n-1}^0) \cup V(Q_{n-1}^1)$, and the edge set is $E(Q_n) = E(Q_{n-1}^0) \cup E(Q_{n-1}^1) \cup M$ where M is a set of edges connecting the vertices of Q_{n-1}^0 and Q_{n-1}^1 in a one to one fashion.

Let n be a positive integer. The n-dimensional star graph, denoted by S_n, is an $(n-1)$-regular graph with $n!$ vertices. The vertex set $V(S_n) = \{v_n v_{n-1} \dots v_1 \mid v_i \in \{1,2,\dots,n\}$ and $v_j \ne v_k$ for $j \ne k\}$. The adjacency is defined as follows: $u_n u_{n-1} \dots u_i \dots u_1$ is adjacent to $v_n v_{n-1} \dots v_i \dots v_1$ through an edge of dimension i with $1 \le i \le n-1$ if $u_j = v_j$ for $j \notin \{i, n\}$, $u_n = v_i$, and $u_i = v_n$. According to the recursively constructed structure of the star graphs S_n, an S_n could be partitioned into n identical $(n-1)$-dimensional star graphs, denoted by $\{S_{n-1}^x \mid 1 \le x \le n\}$, and $V(S_{n-1}^x) = \{v_n v_{n-1} \dots v_2 x \mid v_i \in \{1,2,\dots,n\} \backslash \{x\}$ and $v_j \ne v_k$ for $j \ne k\}$. The attractive features of the star graphs include vertex and edge symmetry, low degree of the vertex, small diameter, surface area, recursive structure, high degree of fault tolerance, and diagnosability [2,9].

In this paper, we shall investigate in the MCDS of hypercubes and star graphs, denoted by MCDS(Q_n) and MCDS(S_n) respectivitly.

3. Minimum Connected Dominating Set of Hypercubes and Star Graphs

According to the recursively constructed structure of the hypercubes Q_n, a Q_n could be partitioned into eight identical $(n-3)$-dimensional hypercubes, denoted by $\{Q_{n-3}^{ijk} \mid i,j,k \in \{0,1\}\}$, and $V(Q_{n-3}^{ijk}) = \{ijk v_{n-3} v_{n-4} \dots v_1 \mid v_i \in \{0,1\}$ for $1 \le i \le n-3\}$. An upper bound of cardinality of the MCDS of hypercubes is shown in the following theorem.

Theorem 1 Given an n-dimensional hypercube Q_n for $n \ge 3$. $|\text{MCDS}(Q_n)| \le 2^{n-2} + 2$.

Proof. To show that $|\mathrm{MCDS}(Q_n)| \le 2^{n-2} + 2$ for $n \ge 3$, a construction scheme is given as follows. Let $i,j,k \in \{0,1\}$ and $\mathrm{CDS}(Q_n) = V(Q_{n-3}^{ijk}) \cup V(Q_{n-3}^{\overline{ijk}}) \cup \{\bar{i}jkv_{n-3}v_{n-4} \dots v_1, \overline{ij}kv_{n-3}v_{n-4} \dots v_1\}$, where $v_x \in \{0,1\}$ for $1 \le x \le n-3$. Then, the induced subgraph $Q_n[\mathrm{CDS}(Q_n)]$ is connected and each vertex v of Q_n is either in $\mathrm{CDS}(Q_n)$ or has at least a neighbor in $\mathrm{CDS}(Q_n)$. Thus, $\mathrm{CDS}(Q_n)$ is a connected dominating set of Q_n and $|\mathrm{CDS}(Q_n)| = 2^{n-2} + 2$. As a result, $|\mathrm{MCDS}(Q_n)| \le 2^{n-2} + 2$ for $n \ge 3$.

Let's take the hypercube Q_5 for an example. Let i=0, j=0, k=0, v_2=0, and v_1=0. Then, $\mathrm{CDS}(Q_5) = V(Q_2^{000}) \cup V(Q_2^{111}) \cup \{10000, 11000\}$, and thus $\mathrm{CDS}(Q_5)$ is a connected dominating set of Q_5 with $|\mathrm{CDS}(Q_5)| = 2^{5-2} + 2 = 10$. As a result, $|\mathrm{MCDS}(Q_5)| \le 10$. See Fig. 2.

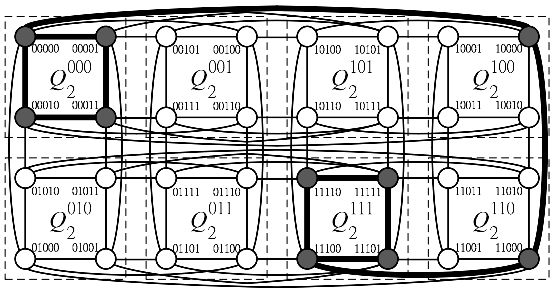

Fig. 2: A CDS of the hypercube Q_5.

For the lower dimensional hypercubes Q_n, n = 3,4,5, we have shown the lower bounds of cardinality of their MCDS that $|\mathrm{MCDS}(Q_n)| \ge 2^{n-2} + 2$ by brute force. Since the proof is tedious and long, we omit it here. By Theorem 1, $|\mathrm{MCDS}(Q_3)| = 4$, $|\mathrm{MCDS}(Q_4)| = 6$, and $|\mathrm{MCDS}(Q_5)| = 10$. Hence, we have the following conjecture.

Conjecture 1 Given an n-dimensional hypercube Q_n for $n \ge 3$. $|\mathrm{MCDS}(Q_n)| = 2^{n-2} + 2$.

An upper bound of cardinality of the MCDS of star graphs is shown in the following theorem.

Theorem 2 Given an n-dimensional star graph S_n for $n \ge 3$. $|\mathrm{MCDS}(S_n)| \le 2(n-1)!$.

Proof. To show that $|\mathrm{MCDS}(S_n)| \le 2(n-1)!$ for $n \ge 3$, a construction scheme is given as follows. Let $x \in \{1,2,\dots,n\}$ and $\mathrm{CDS}(S_n) = \{ v_n v_{n-1} \dots v_2 x \mid v_i \in \{1,2,\dots,n\} \backslash \{x\}$ and $v_j \ne v_k$ for $j \ne k \} \cup \{ x v_{n-1} v_{n-2} \dots v_1 \mid v_i \in \{1,2,\dots,n\} \backslash \{x\}$ and $v_j \ne v_k$ for $j \ne k \}$. Then, the induced subgraph $S_n[\mathrm{CDS}(S_n)]$ is connected and each vertex v of S_n is either in $\mathrm{CDS}(S_n)$ or has at least a neighbor in $\mathrm{CDS}(S_n)$. Thus, $\mathrm{CDS}(S_n)$ is a connected dominating set of S_n and $|\mathrm{CDS}(S_n)| = 2(n-1)!$. As a result, $|\mathrm{MCDS}(S_n)| \le 2(n-1)!$ for $n \ge 3$.

Let's take the star graph S_4 for an example. Let x = 1 and thus $\mathrm{CDS}(S_4) = \{ v_4 v_3 v_2 1 \mid v_i \in \{2,3,4\}$ and $v_j \ne v_k$ for $j \ne k \} \cup \{ 1 v_3 v_2 v_1 \mid v_i \in \{2,3,4\}$ and $v_j \ne v_k$ for $j \ne k \}$. Then, $\mathrm{CDS}(S_4)$ is a connected dominating set of S_4 and $|\mathrm{CDS}(S_4)| = 2(4-1)! = 12$. As a result, $|\mathrm{MCDS}(S_4)| \le 12$. See Fig. 3.

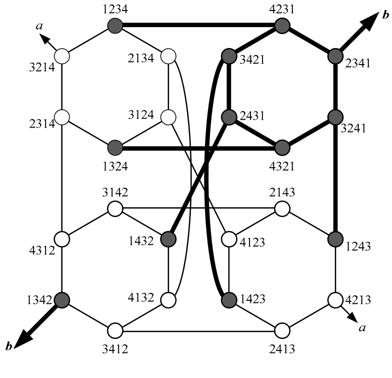

Fig. 3: A CDS of the star graph S_4.

For the lower dimensional star graphs S_n, $n = 3,4$, we have shown the lower bounds of cardinality of their MCDS that $|\text{MCDS}(S_n)| \geq 2(n-1)!$ by brute force. Since the proof is tedious and long, we omit it here. By Theorem 2, $|\text{MCDS}(S_3)| = 4$ and $|\text{MCDS}(S_4)| = 12$. Hence, we have the following conjecture.

Conjecture 2 Given an n-dimensional star graph S_n for $n \geq 3$. $|\text{MCDS}(S_n)| = 2(n-1)!$.

4. Concluding Remarks

The hypercubes Q_n and star graphs S_n are both recursively constructed networks, and they have many attractive properties. This paper demonstrates upper bounds of cardinality of the MCDS of hypercubes and star graphs. For the lower dimensional hypercubes and star graphs, we give lower bounds of cardinality of their MCDS. Therefore, we have conjectures that $|\text{MCDS}(Q_n)| = 2^{n-2} + 2$ and $|\text{MCDS}(S_n)| = 2(n-1)!$ for $n \geq 3$.

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