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Unsteady Flow of a Jeffrey Fluid In An Elastic Tube With A Stenosis

Sreenadh Sreedharamalle [1], Devaki Palluru [1], Diwakar Reddy[2] and Krishnaiah[2]

[1] Department of Mathematics, Sri Venkateswara University, Tirupati- 517 502, A.P. INDIA

[2] Department of Mechanical Engineering, Sri Venkateswara University, Tirupati- 517 502, A.P. INDIA

Abstract. Unsteady flow of a Jeffery fluid in an elastic tube with stenosis has been investigated. The governing equation for the excess pressure is obtained for Jeffrey model. The governing equation has been solved numerically and investigations are made for different cases for Straight, Tapered and Constricted tubes. In the absence of Jeffrey parameter, our results agree with Ramachandra Rao [2] for Newtonian fluid flow in an elastic tube. We find some interesting observations for different parameters on the velocity and excess pressure (the pressure difference between the pressure in the fluid and the external pressure). One of the interesting phenomenon of this paper is that, we studied different types of pressure radius relations, which warrant further study on the non-Newtonian fluid phenomena in elastic tubes.

Keywords: Unsteady Flow, Jeffrey fluid, Elastic tube, Stenosis.

1. Introduction

An elastic material is one that deforms immediately upon loading, maintains a constant deformation as long as the load is held constant, and returns immediately to its original undeformed shape when the load is removed. This module will also introduce two essential concepts in Mechanics of Materials: stress and strain. A material that obeys Hooke's Law is called Hookean. Such a material is elastic according to the description of elasticity and it is also linear in its relation between stress and strain (or equivalently, force and deformation). Therefore a Hookean material is linear elastic, and material engineers use these descriptors interchangeably. Interruption of blood flow in either the arterial or venous system interferes with the delivery of oxygen and nutrients to the tissues. Further, it is realized that unfortunately there are numerous arterial diseases which result in the occlusion of blood flow. An appropriate model will assist in the design of a more accurate method to detect these diseases. Thus, the study of models for blood flow in an elastic tube has became of great interest among many clinicians, physiologists, hemorheologists and mechanical engineers, etc. With this interest many authors are concentrating on the flow of biofluid in an elastic tube.

Rubinow and Keller [1] studied the flow of a viscous fluid through an elastic tube with applications to blood flow. Ramachandra Rao [2] studied on the unsteady flow with attenuation in a fluid filled elastic tube with a stenosis. Oscillatory flow of a viscous fluid in an elastic tube of variable cross section was also investigated by Ramachandra Rao[3]. Sarkar and Jayaraman [4] investigated on nonlinear analysis of oscillatory flow in the annulus of an elastic tube. Srinivas et. al [5] studied the effect of slip, wall properties and heat transfer on MHD peristaltic transport. Influence of wall properties on the peristaltic motion of a Herschel-Bulkley fluid in a channel was studied by Radhakrishnamacharya et. al [6]. More recently Vajravelu et. al [7] studied the flow of a Herschel-Bulkley fluid in an elastic tube. Since the blood is frequently referred as a non-Newtonian fluid, Jeffrey model is preferred by many authors to describe flow of physiological fluids in tubes and channels. Vajravelu et al. [8] studied the influence of heat transfer on peristaltic transport of Jeffrey fluid in a vertical porous stratum and many authors are now concentrating on this Jeffrey model as it is the simplest non-Newtonian fluid model describing some physiological and industrial fluids, [9-11]. Among all these investigations it is observed that not much work is done on the flow of Jeffrey fluid in an elastic tube.

In view of this, the present paper deals with the flow of Jeffrey fluid in an elastic tube with a stenosis. Here we are concentrating on the excess pressure and velocity of the fluid flow. Some interesting observations made for variation of parameters on the velocity and excess pressure.

2. Formulation of the Problem

Consider the flow of blood, which is taken as an incompressible Newtonian fluid, in an elastic tube of circular cross section. We use cylindrical polar coordinates (r,θ,z) with z axis along the axis of the tube and $r=a_0 s(z)$ an arbitrary function of z is the radius of cross section at any axial point z , a_0 being the undisturbed uniform radius of the tube. Under the assumption that the tube cross section varies slowly in the axial direction, we have $\varepsilon = a_0 / L << 1$, where L is a characteristic length along the axis of the tube. It could be the length of the disturbance due to the constriction extending on either side of it or an appropriate wave length. Making use of the unsteady lubrication theory, Ramachandra Rao [3], the equations governing the motion of the fluid in the tube are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial r}+\upsilon[\frac{1}{(1+\lambda_1)}\frac{\partial}{\partial r}(\frac{u}{r}+\frac{\partial u}{\partial r})+\frac{\partial^2 u}{\partial z^2}] \tag{1}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}+\upsilon[\frac{1}{r}\frac{\partial}{\partial r}(\frac{r}{(1+\lambda_1)}\frac{\partial w}{\partial r})+\frac{\partial^2 w}{\partial z^2}] \tag{2}$$

Where u, w are the velocity components in r and z directions respectively, ρ_0 is the density of the fluid, ν is the kinematic coefficient of viscosity and p is the pressure. The equation of continuity is

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0 \tag{3}$$

The other equation for the radial displacement ξ is given by

$$\frac{\partial^2 \xi}{\partial t^2}=\frac{1}{h\rho}(p-2\upsilon\rho_0\frac{\partial u}{\partial r})_{r=a}-\frac{B}{\rho}\frac{\xi}{a^2} \tag{4}$$

Where h and ρ are the thickness and density of the material of the tube and $B=\frac{E}{(1-\sigma^2)}$, E is the Young's modulus and σ is the Poisson's ratio. The boundary conditions for the motion of the tube and the tube are

$$u=\frac{\partial \xi}{\partial t}, \qquad w=0; \qquad \text{on} \qquad r=a_0 s(z) \tag{5}$$

Where a_0 the radius of the tube without elasticity, L is the length of the tube. We assume that $\varepsilon=\frac{a_0}{L}$ is small for a tube with slowly varying cross-section.

The non-dimensional quantities are:

$$\bar{u}=\frac{1}{\varepsilon U_0}u,\quad \bar{w}=\frac{1}{U_0}w,\quad \bar{t}=\omega t,\quad \bar{\xi}=\frac{1}{a_0}\xi,\quad \bar{r}=\frac{1}{a_0}r,\quad \bar{z}=\frac{\varepsilon}{a_0}z,\quad \bar{p}=\frac{\varepsilon a_0}{\rho_0 \upsilon U_0}p \tag{6}$$

Where U0 is the characteristic velocity and ω is the frequency of oscillatory flow.

Introducing the above non-dimensional quantities, neglecting ε and higher order terms completely and based on the assumption that the flow is steady oscillatory, we take

$$(u,w,p,\xi)=e^{it}(\tilde{u},\tilde{w},\tilde{p},\tilde{\xi}) \tag{7}$$

Then Eqs.(1) – (5), become

$$\frac{\partial \tilde{p}}{\partial r}=0 \tag{8}$$

$$\frac{\partial^2 \tilde{w}}{\partial r^2}+\frac{1}{r}\frac{\partial \tilde{w}}{\partial r}+\lambda^2(1+\lambda_1)\tilde{w}=(1+\lambda_1)\frac{\partial \tilde{p}}{\partial z} \tag{9}$$

$$\frac{\partial \tilde{u}}{\partial r}+\frac{\tilde{u}}{r}+\frac{\partial \tilde{w}}{\partial z}=0 \tag{10}$$

$$\tilde{w}=0, \qquad \tilde{u}=iS_t\tilde{\xi}; \qquad on \qquad r=s \tag{11,12}$$

Where $\alpha = a_0\left(\frac{\omega}{\upsilon}\right)^{\frac{1}{2}}$ (Womersley parameter), $R=\frac{U_0 a_0}{\upsilon}$ (Reynolds number), $\tilde{\xi}=\frac{\lambda{*_1}^2}{R{S_t}^2}S^2 p_e$

$S_t=\frac{\omega a_0}{U_0}$ (Strouhal number), $\lambda{*_1}^2=\frac{\omega^2 L^2 \rho_0 a_0}{C^2\rho h}$, $C^2=\frac{B}{\rho}$, $\lambda^2=i\alpha^2$ and p_e is the excess pressure on the wall of the tube. From (13), the pressure is a function of z alone and it is taken as the excess pressure p_e on the wall. The solution (14) satisfying the condition (16) is

$$w=\frac{e^{it}}{\lambda^2}\frac{dp_e}{dz}\left(1-\frac{J_0(\lambda\sqrt{1+\lambda_1}r)}{J_0(\lambda\sqrt{1+\lambda_1}S)}\right) \tag{13}$$

Where $\lambda^2=-i\alpha,$ $J_n(z)$ is a Bessel function of first kind order n, and $p_e=p-p_0$ is the excess pressure, p_0 is the undisturbed pressure with its hydrostatic distribution, is a function of z alone. The flux across any cross section of the tube is given by

$$Q=2\pi\int_0^S wrdr=-\frac{\pi}{\lambda^2}e^{it}\frac{dp_e}{dz}S^2\frac{J_2(\lambda\sqrt{1+\lambda_1}S)}{J_0(\lambda\sqrt{1+\lambda_1}S)} \tag{14}$$

The non-dimensional equation of continuity for longitudinal motions influenced by area changes, Lighthill [12], is

$$\frac{wL}{U_0}\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial z}=0 \tag{15}$$

Where A is the cross sectional area at any axial point of the tube. The elastic tube is assumed to be tethered against longitudinal displacements. In general the pressure p at any point is a function of area A (z, t), z and time t. The elasticity of the wall is introduced into the problem by a relation between the excess pressure $p-p_0$ the transmural pressure of an artery, and area A which is assumed to be known. Such pressure-radius relations for thick and thin walled tubes satisfying the boundary conditions at deformed or undeformed points have been listed by Taylor and Gerrard [13] and Kurz [14]. Further we know

We know that

$$\frac{\partial p}{\partial t}=\frac{\partial p}{\partial A}\frac{\partial A}{\partial t} \tag{16}$$

$$C^2(z)=\frac{A}{Q_0}\frac{\partial p}{\partial A}=\frac{S}{2Q_0}\frac{\partial p}{\partial S} \tag{17}$$

Where c(z) is the local value of the wave speed in the tube and q_0 is the density of the fluid, Lighthill [12]. From (22) and (23), we have

$$\frac{\partial A}{\partial t}=\frac{A}{C^2}\frac{\upsilon U_0 L}{a_0 z}ie^{it}p_e \tag{18}$$

Using (20) and (24) in (21), we get

$$\frac{d^2 p_e}{dz^2}+\frac{2}{S}\frac{dS}{dz}a_1(\lambda\sqrt{1+\lambda_1}S)\frac{dp_e}{dz}-\frac{w^2 l^2}{C^2}a_2(\lambda\sqrt{1+\lambda_1}S)p_e=0 \tag{17}$$

Where $a_1(\lambda\sqrt{1+\lambda_1}S)=\frac{J_1^2(\lambda\sqrt{1+\lambda_1}S)}{J_0(\lambda\sqrt{1+\lambda_1}S)J_2(\lambda\sqrt{1+\lambda_1}S)}$, $a_2(\lambda\sqrt{1+\lambda_1}S)=\frac{J_0(\lambda\sqrt{1+\lambda_1}S)}{J_2(\lambda\sqrt{1+\lambda_1}S)}$

Ramachandra Rao [3] has derived an equation similar to (25) using shell equations for a thin walled elastic tube and is given by

$$\frac{d^2 p_e}{dz^2} + \frac{2}{S} a_1 \frac{dS}{dz}\frac{dp_e}{dz} - 2S\lambda *_1^2 a_2 p_e = 0 \tag{18}$$

Where, $\lambda *_1^2 = \frac{w^2 L^2 \rho_0 a_0}{C_0^2 \rho h}$, ρ is the density of the tube, h is the thickness of the tube, $C_0^2 = \frac{B}{\rho}, B = E/(1-\sigma^2)$, E being the Youngs modulus and σ is the Poisson ratio. Further we may write

$$\lambda_a^2 = 2\lambda *_1^2 = \frac{\omega^2 L^2}{C_1^2} \tag{19}$$

Where $c_1^2 = Bh/2\rho_0 a_0$ is the classical Moens-Korteweg speed, Pedley [15]. We list below few of the pressure-radius relations for thin walled elastic tubes, these have been used in numerical work afterwards, given by and Taylor and Gerrard [13] and Kurz [14]

(i) $p = \rho_0 c_1^2 (1 - \frac{1}{s})$ (ii) $p = \frac{8}{3}\rho_0 c_1^2 (1 - \frac{1}{s^2})$ (iii) $p = 2\rho_0 c_1^2 \frac{1}{s^2}(s-1)$ (iv) $p = \frac{8}{3}\rho_0 c_1^2 \frac{1}{s^2}(s-1)$
(v) $p = \frac{2}{3}\rho_0 c_1^2 (1 - \frac{1}{s^4})$

It can be easily observed that Eq. (27) can be derived from (25) by substituting the expression for c^2 as calculated from (8) using the pressure-radius relation (i). Thus Eq. (27) for excess pressure is more general and is valid for a wider range of conditions. Further, it is clear from the pressure-radius relationships listed that the local value of the wave speed $c^2(z)$ at any axial station is expressible as $c_1^2 / F(s)$ where $F(s)$ is a function of s depending upon the pressure-radius relation chosen. The Eq. (10), in general is written in the form

$$\frac{d^2 p_e}{dz^2} + \frac{2}{S} a_1 \frac{dS}{dz}\frac{dp_e}{dz} - \lambda_a^2 F(S) a_2 p_e = 0 \tag{20}$$

Where the parameter λ_a defined in (29) is a ratio of ωL, the velocity coming through the flow and geometry to the Moens-Korteweg speed. It appears that λ_a plays crucial role in these types of problems and the importance of this parameter is still to be established through experiments. The approximate equation for excess pressure for Womersley parameter large is given by

$$\frac{d^2 p_e}{dz^2} + \frac{2}{S}\frac{dS}{dz}\left(1 - \frac{i}{\lambda\sqrt{1+\lambda_1 S}}\right)\frac{dp_e}{dz} - \lambda *_1^2 F(S)\left(1 - \frac{2i}{\lambda\sqrt{1+\lambda_1 S}}\right) p_e = 0 \tag{21}$$

The excess pressure given by (36) is complex, by writing $p_e = p_r + ip_i$, and equating real and imaginary parts, we obtain two coupled ordinary differential equations of second order for p_r and p_i. These equations are rewritten as four first order equations and are solved using Mathematica by prescribing the initial conditions at some point z of the axial section of the tube. In our problem we have chosen the initial point as z = 1 and the upper boundary for z as 10 for all the geometries of the tubes considered. The modulus of pressure $|p_e|$ has been evaluated for a) Straight tube given by $S = 1, 1 < z < 10$, b) Tapered tube given by $S(z) = \exp(-0.025z), 1 < z < 10$, c) Locally constricted tube given by $S = 1 - 0.5\exp(-(z-6)^2), 1 < z < 10$.

The initial conditions are taken as

$$p_r = 0.1, \quad p_i = 0, \quad \frac{dp_r}{dz} = 0, \quad \frac{dp_i}{dz} = -0.01 \quad at \quad z = 0 \tag{22}$$

3. Results and Discussions

We study the flow of a Jeffrey fluid with stenosis in an elastic tube. Here the effects of Jeffrey parameter, external pressure, pressure radius relations and the elastic nature of the tube are explained in detailed as follows.

From Fig - (1) we observe that as the Jeffrey parameter increases , the effect of external pressure in decreasing for a tapered tube. Fig - (2) shows that for an increase in the Jeffery parameter, there is a decrease in the effect of external pressure for a linear tube. If the tube is locally constricted, we notice that as the Jeffrey parameter increases, the effect of external pressure on the fluid flow is decreasing, which is shown in Fig - (3). Fig - (4) and Fig - (5) shows the variation of external pressure with z for different pressure radius relations (i), (ii), (iii), (iv) and (v) for a locally constricted tube and tapered tube respectively. If the pressure radius relation is of the type (i), the external pressure will be high for both the locally constricted tube and for a tapered tube i.e. the high external pressure gives rise to a low flow rate and if the pressure radius relation is of the type (v) the flow rate will be more due to low external pressure for both the locally constricted tube and for a tapered tube.

From Fig - (6), we observe that flux increases as the Jeffrey parameter increases. We notice from Fig - (7) that as the Jeffrey parameter increases the velocity is increasing. Figs. (8, 9 & 10) shows the variation of external pressure with z for different values of λ_a for a tapered tube, Locally Constricted and for a linear tube respectively. We notice from these graphs that as λ_a increases, the pressure oscillates more. In particular if λ_a takes the values 0.707 and 1.414, there is not much change in the amplitude of the pressure propagation for a tapered tube and for a linear tube. Also there is a change in the magnitude of the pressure propagations for the tree types of the tubes. But it is observed that there is not much change in the external pressure wave propagations if λ_a takes the values 0.1 and 0.2 for all the three different types of tubes.

4. Conclusions

We study the flow of a Jeffrey fluid with stenosis in an elastic tube. Here the effects of Jeffrey parameter, external pressure, pressure radius relations and the elastic nature of the tube are explained in detailed as follows.

- As the Jeffrey parameter increases, the effect of external pressure in decreasing for a tapered tube.
- Increase in the Jeffery parameter, there is a decrease in the effect of excess pressure for a linear tube.
- If the tube is locally constricted, we notice that as the Jeffrey parameter increases, the effect of external pressure on the fluid flow is decreasing.
- If the pressure radius relation is of the type (i), the external pressure will be high for both the locally constricted tube and for a tapered tube i.e. the high

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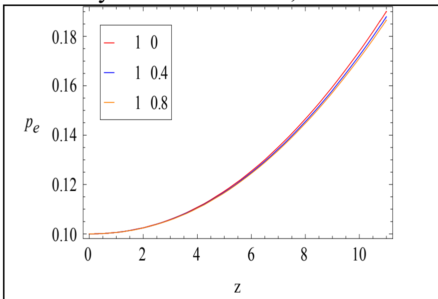

Fig 1: Variation of external pressure with z for different values of the Jeffrey parameter λ_1, for a tapered tube

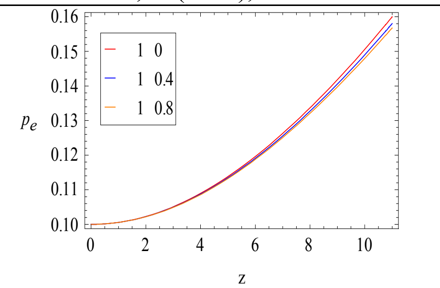

Fig 2: Variation of external pressure with z for different values of the Jeffrey parameter λ_1, for a straight tube

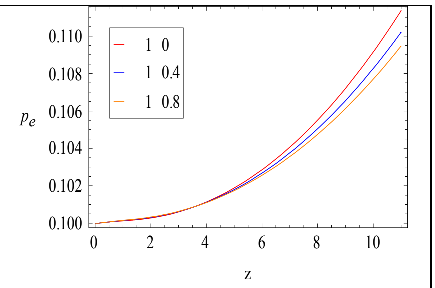

Fig 3: Variation of external pressure with z for different values of the Jeffrey parameter λ_1, for a locally constricted tube

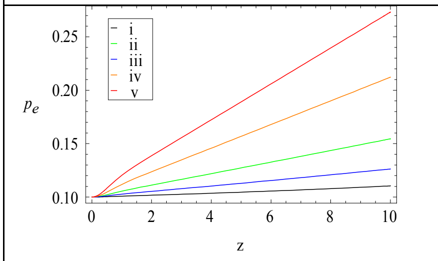

Fig 4: Variation of external pressure with z for different pressure-radius relations for a locally constricted tube

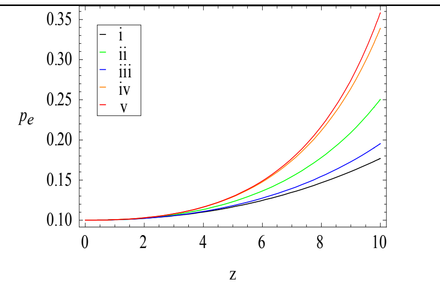

Fig 5: Variation of external pressure with z for different pressure-radius relations for a tapered tube

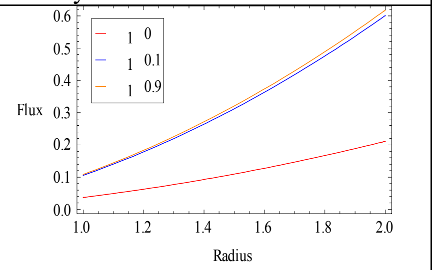

Fig 6: Variation of flux with radius for different Jeffrey parameters

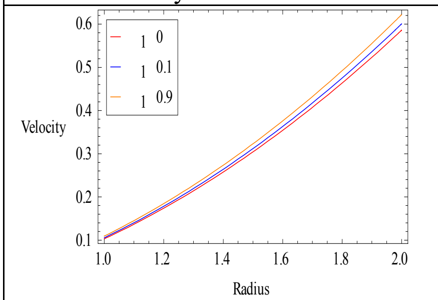

Fig 7: Variation of velocity with radius for different Jeffrey parameters.

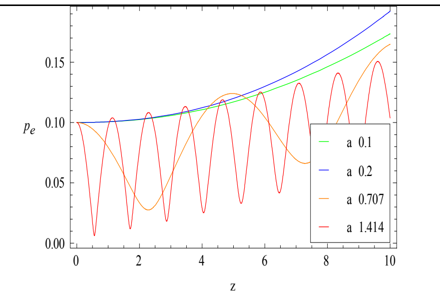

Fig 8: Variation of external pressure with z for different values of elastic parameter for a tapered tube

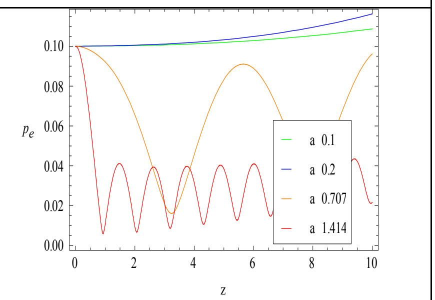

Fig 9: Variation of external pressure with z for different values of elastic parameters for a locally constricted tube

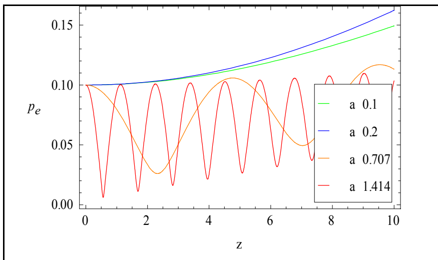

Fig 10: Variation of external pressure with z for different values of elastic parameters for a linear tube.

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