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Double diffusive Rotating convection In a sparsely packed Porous Medium

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Abstract

Linear stability analysis of Double diffusive rotating convection in a sparsely packed Porous medium with free-free boundary conditions has been investigated by Chandrasekhar [15] who determined the marginal stability boundary and critical wave numbers for the onset of convection and over stability as a function of the Chandrasekhar number. No closed form formulae appeared to exist and the results were tabulated numerically. However, by taking the Rayleigh number as dependent and independent variable we have found remarkably simple expressions.

Keywords: Convection, Bifurcation points, Boussinesq approximation.

1 Introduction

Nonlinear rotating convection in a porous medium uniformly heated from below is of considerable interest in geophysical fluid dynamics, as this phenomena may occur within the Earth's outer core. Earth's outer core consists of molten Iron and lighter alloying element, sulphur in its molten form. This lighter alloying element present in the liquid phase is released as the new iron freezes due to supercooling onto the solid Inner core. Hence we get mushy layer near the inner core boundary where the problem becomes convective instability in a porous medium Roberts et. al. [12]. The mushy layer is a region of coexisting liquid and solid phases, forming as a consequence of constitutional supercooling, when a binary alloy solidifies directionally Worster [8].

Rotating convection in an electrically conducting fluid in a nonporous medium has been studied extensively (Chandrasekhar [15], Buell and Catton [7], Tagare et.al., [16, 17, 18], Guba and Bod'a [11] and Guba [10]). But its counter part in a porous medium has not been given much attention inspite of its geophysical applications. Palm et. al., [2] investigated Rayleigh-Benard convection problem in a porous medium. Brand and Steinberg [5] investigated convecting instabilities in binary liquid in a porous medium. However, Palm et. al., [2] and Brand and Steinberg [6], Steinberg and Brand [19] have made use of Darcy's law ($-\nu\nabla^2\overline{V}$ is replaced by $K\overline{V}$ where K is the permeability of a porous medium. For nonporous medium K is infinity). They have also not considered usual convective nonlinearity. It is well known that Darcy's law breaks down in situations where in other effects like viscous shear and inertia come into play. In fact Darcy's law is applicable to densely packed porous medium. An alternative to Darcy's equation is Brinkman equation and is of the form

$$\nabla'\rho' - \rho'\overline{g} = -\frac{\mu}{K}\overline{V}' + \mu_e\nabla'^2\overline{V}',$$

where μ is the fluid viscosity and μ_e is the effective fluid viscosity, Brinkman model is valid for a sparsely packed porous medium wherein there is more window fluid to flow so that the distortion of velocity give rise to the usual shear force. Lapwood [3] was the first to suggest the inclusion of convective term $(\overline{V}'.\nabla')\overline{V}'$ in the momentum equation and study the Rayleigh-Benard convection in a sparsely packed porous medium. Recently, Tagare and Benerji Babu [16] have investigated the problem of nonlinear convection in a sparsely packed porous medium due to thermal and compositional buoyancy.

In this paper we investigate the problem of Double diffusive rotating convection in a sparsely packed porous medium. Rudraiah and Srimani [9] have studied linear stability analysis in the case of thermal convection in a rotating fluid saturated porous medium using Brinkman model but they have taken effective viscosity μ_e same as fluid viscosity μ, However, experiments show that the ratio of effective viscosity μ takes the value ranging from 0.5 to 10.9 Givler and Altobelli [13]. In Section 2, we write basic dimensionless equations in Boussinesq approximation for Double diffusive rotating convection a in a sparsely packed medium by using for a momentum equation Darcy-Lapwood-Brinkman model with effective viscosity different from fluid viscosity. In Section 3, we study linear stability analysis by considering R as dependent and independent variable for instance Kloosterziel [14]. In Section 4, we write conclusions of the paper.

2 Basic Equations

Consider a horizontal, infinitely extended layer of fluid in a porous medium of depth d which is kept rotating at a constant angular velocity Ω about z-axis, this layer is heated from below. The upper and lower bounding surfaces of the layer are assumed to be stress-free. Physical properties of the fluid are assumed constant, except density in the buoyancy term, so that the Boussinesq approximation is valid. The porous medium is considered homogeneous and isotropic. The onset of convection is such a layer is governed by the following equations (Neild and Bejan [1])

$$\nabla.\overline{V} = 0, \tag{1}$$

$$\frac{1}{M^2\phi Pr}\left[\frac{\partial \overline{V}}{\partial t} + \frac{1}{\phi}(\overline{V}.\nabla)\overline{V}\right] = -\nabla\left(\frac{P}{MPr} - \frac{TaPr}{8M\phi^2}|\hat{e}_z \times \overline{r}|^2\right) - \frac{1}{MD_a}\overline{V} + \frac{Ta^{\frac{1}{2}}}{\phi}\left(\overline{V}\times\hat{e}_z\right) + \frac{\Lambda}{M}\nabla^2\overline{V} + R\theta\hat{e}_z, \tag{2}$$

$$\frac{\partial \theta}{\partial t} + \frac{1}{M}(\overline{V}.\nabla)\theta = \frac{w}{M} + \nabla^2\theta. \tag{3}$$

The dimensionless numbers required for the description of the motion are Rayleigh number: $R = g\alpha\Delta T d^3/\kappa\nu$, Prandtl number $Pr = \nu/\kappa_T$, Darcy number $D_a = \kappa_T/d^2$ and Taylor number $Ta = 4\Omega^2 d^4/\nu^2$.

It is convenient to reduce equations (1)-(3) to a single equation for the vertical velocity V by eliminating pressure and temperature from the linear differential operator :

$$\mathcal{L}w = \mathcal{N}, \tag{4}$$

3 Linear Stability Analysis

We perform a linear stability analysis of the problem by substituting

$$w = W(z)e^{iqx+pt}, \tag{5}$$

into linearized version of equation (4) is $\mathcal{L}w = 0$, and obtaining an equation

$$\left[(D^2-q^2)(p-D^2+q^2)\left[\frac{p}{M^2\phi Pr} + \frac{1}{MD_a} - \frac{\Lambda}{M}(D^2-q^2)\right]^2 + \frac{Ta}{\phi^2}D^2(p-D^2+q^2) + \frac{Rq^2}{M}\left[\frac{p}{M^2\phi Pr} + \frac{1}{MD_a} - \frac{\Lambda}{M}(D^2-q^2)\right]\right]W(z) = 0, \tag{6}$$

where $D = (d/dz)$. We consider stress-free boundary conditions, then $W = D^2W = 0$ on $z = 0$, $z = 1$ for all x, y.

3.1 Determination of marginal stability when Rayleigh number R is a dependent variable

Substituting $W(z) = \sin \pi z$ and $p = i\omega$ into (6), we get

$$R = \frac{M}{q^2}\left[A_1 + i\omega(A_2\omega^2 + A_3)\right], \tag{7}$$

(a) Stationary Convection ($\omega = 0$)

Substituting $\omega = 0$ in equation (7), we get

$$R_s = \frac{M}{q_s^2}\left[\frac{\delta_s^4\left(\frac{1}{MD_a} + \frac{\Lambda}{M}\delta_s^2\right)^2 + \frac{Ta}{\phi^2}\pi^2\delta_s^2}{\left(\frac{1}{MD_a} + \frac{\Lambda}{M}\delta_s^2\right)}\right], \tag{8}$$

where $\delta_s^2 = (\pi^2 + q_s^2)$. Here R_s is the value of the Rayleigh number for stationary convection. The minimum value of R_s is obtained for $q_s = q_{sc}$, where

$$2\left(\frac{q_{sc}}{\pi}\right)^6 + 3\left(\frac{q_{sc}}{\pi}\right)^4 = 1 + \frac{Ta}{\pi^4}\left(\frac{M}{\Lambda\phi}\right)^2. \tag{9}$$

Threshold for the onset of stationary convection is given by equation (8) with $q_s = q_{sc}$,

$$R_{sc} = \frac{M}{q_{sc}^2}\left[\frac{\delta_{sc}^4\sigma_1^2 + \frac{Ta}{\phi^2}\pi^2\delta_{sc}^2}{\sigma_1}\right], \tag{10}$$

The corresponding asymptotic values of q_{sc} and R_{sc} are

$$q_{sc} \simeq \left(\frac{\pi^2 TaM^2}{2\Lambda^2\phi^2}\right)^{\frac{1}{6}}, \tag{11a}$$

$$R_{sc} \simeq 3\Lambda\pi^4\left(\frac{M^2 Ta}{2\Lambda^2\phi^2\pi^4}\right)^{\frac{2}{3}}. \tag{11b}$$

In the free-free boundary conditions, for large Taylor number, we have

$$R_{sc} \propto Ta^{\frac{2}{3}} \text{ and } q_{sc} \propto Ta^{\frac{1}{6}}. \tag{12}$$

This is also true for rigid-rigid and rigid-free boundary conditions.

(b) Oscillatory Convection ($\omega^2 > 0$)

For the oscillatory convection ($\omega \neq 0$) and from equation (7), R will be complex. But the physical meaning of R requires it to be real. The condition that R is real implies that imaginary part of equation (7) is zero, i.e.,

$$A_2\omega^2 + A_3 = 0, \tag{13}$$

Substituting ω^2 from equation (13) into the real part of equation (7), we get

$$R_o = \frac{2\Lambda(1 + MPr\Lambda\phi)}{Mq_o^2}\left[\delta_o^6 + \frac{M^4\pi^2 Pr^2 Ta}{(1 + MPr\Lambda\phi)^2}\right]. \tag{14}$$

The corresponding asymptotic behavior of q_{oc} and R_{oc} for large Taylor number are (Chandrasekhar [15])

$$q_{oc} \simeq \left(\frac{TaPr^2M^4\pi^2}{2(1 + MPr\Lambda\phi)^2}\right)^{\frac{1}{6}}. \tag{15a}$$

$$R_{oc} \simeq \frac{2\pi^4\Lambda}{M}(1 + MPr\Lambda\phi)\left[2\left(\frac{TaPr^2M^4}{2\pi^4(1 + MPr\Lambda\phi)^2}\right)\right]^{\frac{2}{3}}. \tag{15b}$$

From equations (11b) and (15b), $R_{oc} \to R_{sc}$ as $Ta \to \infty$ implies that for large Taylor number

$$\frac{2M^{\frac{8}{3}}Pr^{\frac{4}{3}}}{(1+MPr\Lambda\phi)^{\frac{1}{3}}} = 1. \tag{16}$$

Root of equation (16) is (Chandhrasekhar [15]) $Pr = Pr_c = 0.783813$. Thus $R_{oc}(q_{oc}) \to R_{sc}(q_{sc})$ at $Pr = Pr_c$. From the monotonic dependence of q_{oc} and q_{sc} on Ta, we may conclude that for $Pr > Pr_c$, $R_{oc} > R_{sc}$ for all Ta.

Equation (10) shows the stabilizing or inhibiting effect of rotation at the onset of stationary convection. The increase of R_{sc} and R_{oc} with the Taylor number Ta implies that disturbances in the fluid will not move upward or downward easily due to the presence of Coriolis force. We have neglected the effect of centrifugal force. For inviscid fluid the Taylor number Ta is infinite and consequently the critical Rayleigh number R_{sc} for the onset of stationary convection in a rotating fluid. Inviscid fluid with rotation is stable for all vertical temperature gradients. This is a consequence of the Taylor -Proudman theorem. The patterns of convection in the presence of rotation depend on both horizontal co-ordinates and Taylor number. An infinite number of patterns are theoretically possible at same critical Rayleigh number. The patterns can be rolls, square cells, rectangular cells of all side ratios and hexagonal cells. Chandrasekhar [15] calculated the velocity fields for these various patterns. Küppers and Lortz [4] showed that with rotation and under slightly supercritical conditions all three-dimensional convective flows are unstable.

3.2 Determination of marginal stability when Rayleigh number R is an independent variable

Putting $W = \sin \pi z$, into equation (6) we get a third degree polynomial equation in p of the following form:

$$p^3 + Bp^2 + Cp + D = 0, \tag{17}$$

The system will be stable when three roots of cubic equation (17) have $Re(p) < 0$. If $Re(p) > 0$ for atleast one root of the cubic equation then the system will be unstable. With each root of the cubic equation there is an associated combination of a flow field and temperature distribution. The instability can set in as stationary convection if one root of the equation (17) is zero or oscillatory convection if two roots are purely imaginary (Kloosterziel and Cornevale [14]).

(a) Stationary Convection ($\omega = 0$):

The stability of the system is determined by the sign of D and $BC - D$. The analytical expressions for critical wave number and critical Taylor number given below

$$q_{sc} = \left[\left(\frac{R}{3\Lambda}\right)^{\frac{1}{2}} - \pi^2\right]^{\frac{1}{2}}, \quad Ta = Ta_{sc} = R\left[\frac{\Lambda^{\frac{1}{2}}\phi^2}{M^2}\left(\frac{R}{R_{rb}}\right)^{\frac{1}{2}} - \frac{\Lambda\phi^2}{M^2}\right] \text{ where } R_{rb} = \frac{27\pi^4}{4}. \tag{18}$$

Here R_{rb} is the critical Rayleigh number for the onset of stationary convection of Rayleigh-Benard convection without rotation. Fig. 2b is plotted in (R, Ta)-plane for the curve (18). In this figure $Ta_{sc} = 0$ on R-axis. From R-axis the curve (18) starting from $R = R_{rb}$. In (R, Ta)-plane we check the sign of D in a range of R with $q = q_{sc}(R)$ at a fixed $Ta = Ta_{sc}$. For the values $\{R, Ta\}$ which are left to the curve (18). $D > 0$ and $D < 0$ for the values $\{R, Ta\}$ which are right to the curve (18) and $D = 0$ on the curve (18).

(b) Oscillatory Convection ($\omega^2 > 0$):

The condition $D > 0$ is not enough to discuss the stability of the system. So we have also to check the sign of $BC - D$ along with the sign of D for the stability of the system. We write the the critical wave number q_{oc} and critical Taylor number Ta_{oc} at the onset of oscillatory convection as

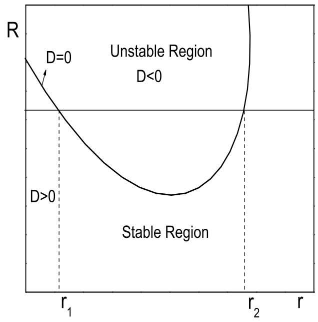

Figure 1: A typical diagram showing the stability regions of the system for stationary convection.

$$q = q_{oc}(R, Pr, \Lambda, \phi, M) = \left[\left(\frac{MR}{6\Lambda(1 + MPr\Lambda\phi)} \right)^{\frac{1}{2}} - \pi^2 \right]^{\frac{1}{2}},$$

$$Ta = Ta_{oc}(R, Pr, \Lambda, \phi, M) = R \left[\left(\frac{(1 + MPr\Lambda\phi)}{2^3 Pr^4 M^5 \Lambda^3} \right)^{\frac{1}{2}} \left(\frac{R}{R_{rb}} \right)^{\frac{1}{2}} - \frac{(1 + MPr\Lambda\phi)}{2Pr^2 M^3 \Lambda} \right]. \quad (19)$$

The explicit expressions near CTP are obtain

$$R = R_{ct} = (1 + \Upsilon)^2 R_{rb}, \; Ta_{ct} = \frac{\Lambda^{\frac{1}{2}} \phi^2}{M^2} (1 + \Upsilon - \Lambda^{\frac{1}{2}})(1 + \Upsilon)^2 R_{rb},$$

$$\Upsilon = \frac{2^{\frac{1}{2}}(1 + MPr\Lambda\phi) - (M(1 + MPr\Lambda\phi))^{\frac{1}{2}}}{(M(1 + MPr\Lambda\phi))^{\frac{1}{2}} - 2^{\frac{3}{2}} Pr^2 \Lambda^2 \phi^2 M^2}. \quad (20)$$

The suffix *ct* in equation (20) stands for parameter at codimension two bifurcation point. At Ta_{ct}, $Ta_{sc} = Ta_{oc}$ and $q_{sc} \neq q_{oc}$. 4(a-d) when Pr increases, the intersection point $\{R_{ct}, Ta_{ct}\}$ moves upwards. 4(a-d), the frequency ω^2 changes its sign from negative to positive before the intersection point as R increases. When $Ta < Ta_{ct}$ we get stationary convection as a first instability while for $Ta > Ta_{ct}$ the first instability will be oscillatory convection.

4 Conclusions

In this paper we have considered both linear stability of Double diffusive rotating convection in a sparsely packed porous medium in Earth's outer core by using free-free (stress-free) boundary conditions. Even though free-free boundary conditions can not be achieved in laboratory, one can use it in geophysical fluid dynamic applications to Earth's outer core since they allow simple trigonometric eigenfunctions. Our goal is to identify the region of parameter values, for which roll emerge at the onset of convection.

We have revisited linear problem of Double diffusive Rotating convection In a sparsely packed Porous Medium with so called free-free boundary conditions. Chandrasekhar [15] described the stationary convection and oscillatory convection in the $RTa-$plane as curves $R_s(Ta, M, \Lambda, \phi)$ and $R_o(Ta, Pr, M, \Lambda, \phi)$ respectively. By considering Rayleigh number R as the dependent and independent variable, we have found in section 3 simple expressions for the stationary convection curve $Ta_{sc}(R, M, \Lambda, \phi)$ and for oscillatory convection $Ta_{oc}(R, Pr, M, \Lambda, \phi)$, Similarly, the critical horizontal wave number for the onset of stationary convection convection $q_{sc}(R, M, \Lambda, \phi)$ and for the onset of oscillatory convection $q_{oc}(R, Pr, M, \Lambda, \phi)$.

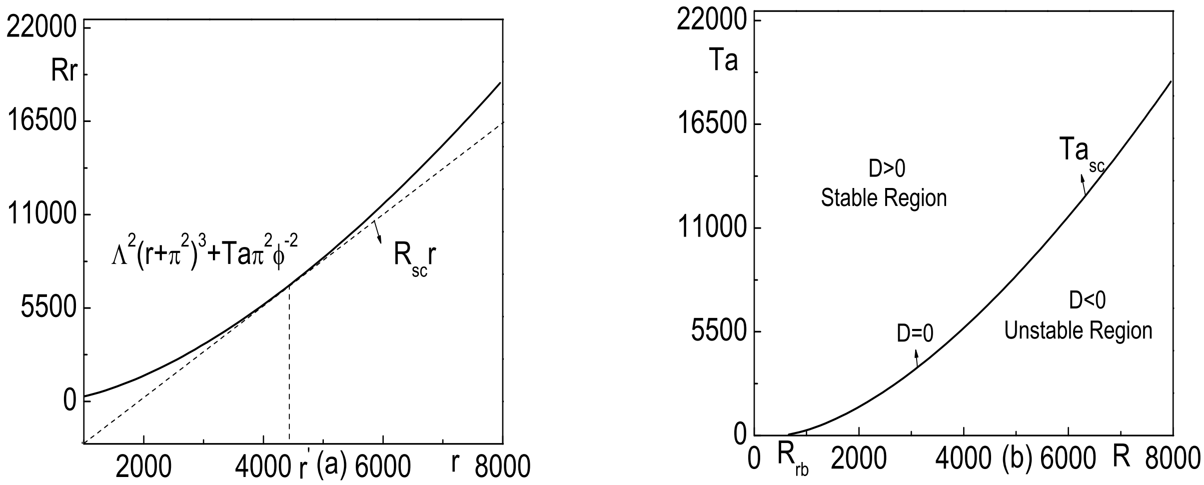

Figure 2: At the intersection point of the curve and straight line in figure (a) we get the critical wave number $q_{sc} = r'^{\frac{1}{2}}$ corresponding to the critical Rayleigh number at a given Taylor number. In figure (b) the system (stationary convection) is stable in $D > 0$ region, unstable $D < 0$ region and $D = 0$ on the curve Ta_{sc}.

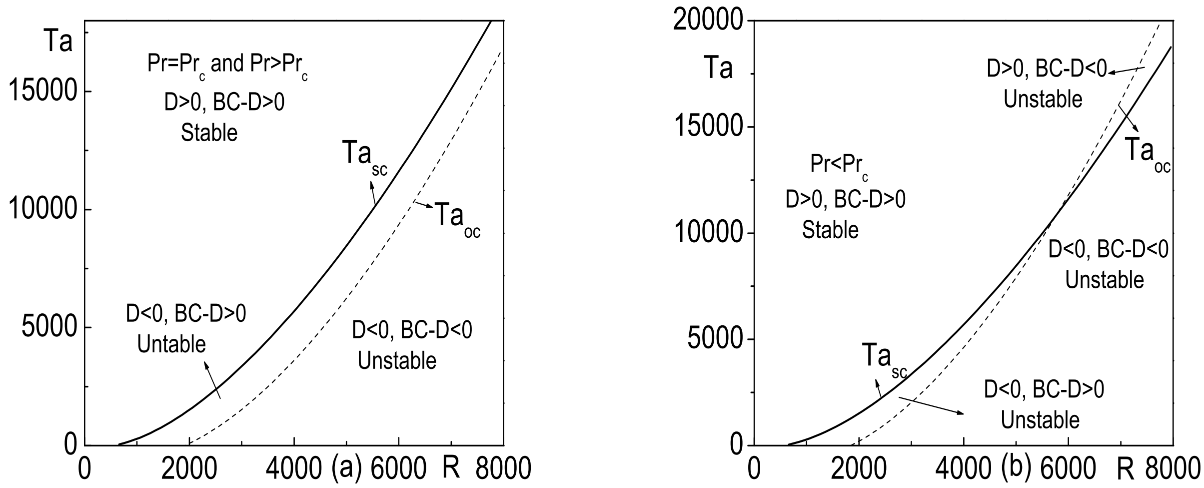

Figure 3: The typical diagram show the stability regions of the system on the solid lines $D = 0$ and on the dotted lines $BC - D = 0$. In each figure for the values $\{R,\ Ta\}$ which are left to the solid line $D > 0$ and $D < 0$ for the values $\{R,\ Ta\}$ which are left to the dotted line and $BC - D < 0$ for the values $\{R,\ Ta\}$ which are right to the dotted line.

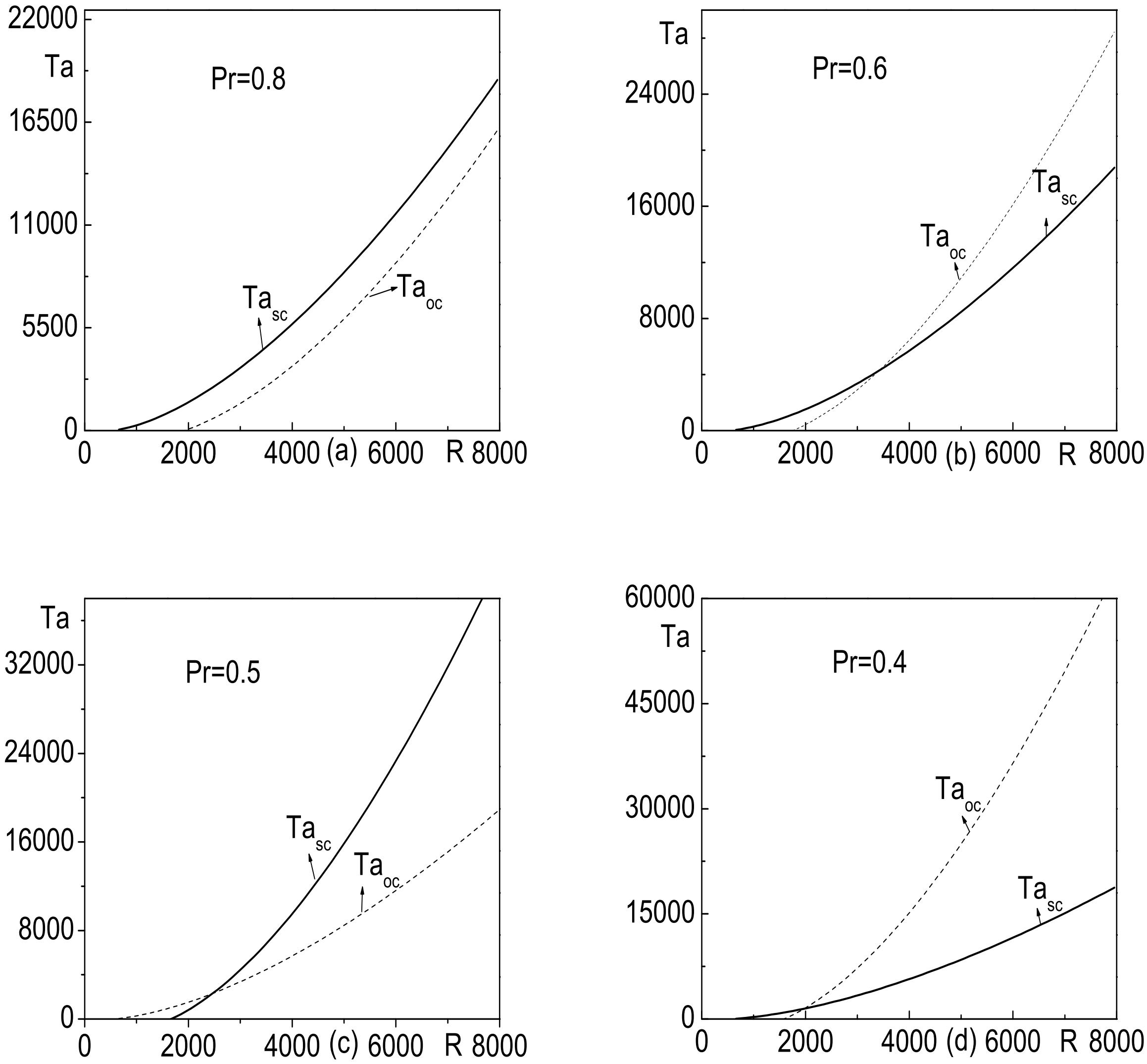

Figure 4: In above figures solid lines are plotted for the curve Ta_{sc} (stationary convection) and dotted lines are plotted for the curve Ta_{oc} (oscillatory convection) at different values of Pr. When $Pr \to 0$ then the intersection point appear at $R_{ct} = \frac{2}{M} R_{rb}$ and $Ta_{ct} = 2M^{\frac{-3}{2}}(2^{\frac{1}{2}} - M^{\frac{1}{2}})R_{rb}$, where as for $Pr \to Pr_c$, $R_{ct} \to \infty$ and $Ta_{ct} \to \infty$.

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