2012 International Conference on Fluid Dynamics and Thermodynamics Technologies (FDTT 2012)
IPCSIT vol. 33 (2012) © (2012) IACSIT Press, Singapore

Particular Solutions to Modified Shallow Water Equations

Bibekananda Bira, T. Raja Sekhar

Department of Mathematics, National Institute of Technology Rourkela-8, India

Abstract. In this paper, we obtain the exact solution to non-linear form of first order partial differential equations (PDEs) which governs one dimensional modified shallow water equations. Lie group analysis is used to identify generators that leave the given system of PDEs invariant. Using these generators, two commuting generators are constructed. With the help of canonical variables associated with these two generators, the original system of PDEs reduced to an autonomous system. Though the autonomous system admit constant solutions for some variables, it may corresponds to non-constant solutions of the original system.

Keywords: Shallow water equations, Group theoretic method, Exact solution, Symmetry generators, Similarity solutions

1. Introduction

We use the symmetry reduction method based on Lie group theory to obtain some exact solutions, so called invariant solutions of a modified shallow water equations. Here, by exact solution we mean one obtained by resolving a reduced system which is reduction of the full set of PDEs of the original system to set of autonomous PDEs. Indeed, for non-linear systems involving discontinuities such as shocks, we do not have the luxury of complete exact solutions, and for analytical work we have to rely on some approximate analytical or numerical methods which may be useful to set the scene and provide useful information towards our understanding of complex physical phenomena involved. One of the most powerful method to determine particular solutions to PDEs is based upon the study of their invariance with respect to the one-parameter Lie group of point transformations [1, 2, 3]. Here we characterized class of solutions of the system by introducing some transformations that map the equations at hand to an equivalent autonomous form [4, 5]. A different approach has been derived by Oliveri and Speciale [6, 7], for unsteady equations of perfect gases and ideal magnetogasdynamics equations using substitution principles. Ames and Donato [8] obtained solution for the problem of elastic-plastic deformation generated by torque, and analysed evolution of weak discontinuities in a state characterized by invariant solutions. Lie group analysis for one layer shallow water equations have been discussed in [9, 10]. Raja Sekhar and Sharma [11] discussed the evolution of weak discontinuities in classical shallow water equations. In this paper, we obtain some exact particular solutions, to the system of PDEs governing one dimensional modified shallow water equations, by using Lie group analysis.

2. Group Analysis

We consider the system of equations which governs the one dimensional modified shallow water equations as follows [12]

$$\boldsymbol{h_t + uh_x + hu_x = 0},$$

$$\boldsymbol{u_t + \frac{g(h+H)}{h} h_x + uu_x = 0} \tag{1}$$

where u is the x-component of fluid velocity, h the water depth, g acceleration due to gravity and $H = k_0/g$ the reduced factor characterizing advective transport of impulse with positive k_0. The independent variables x and t denote space and time respectively. We consider a one parameter Lie group of infinitesimal transformations

$$\tilde{t} = t + \varepsilon\xi^{t}(t,x,\rho,u;\varepsilon) + O(\varepsilon^2), \qquad \tilde{x} = x + \varepsilon\xi^{x}(t,x,\rho,u;\varepsilon) + O(\varepsilon^2)$$
$$\tilde{h} = h + \varepsilon\eta^{h}(t,x,\rho,u;\varepsilon) + +O(\varepsilon^2), \qquad \tilde{u} = u + \varepsilon\eta^{u}(t,x,\rho,u;\varepsilon) + O(\varepsilon^2). \qquad (2)$$

where the generators ξ^t, ξ^x, η^h and η^u which are to be determined in such a way that the partial differential equation (1) are invariant with respect to the transformations (2); the entity ε^2 is so small that its square and higher powers may be neglected. By using Lie group analysis [3], we obtain.

$$\xi^t = a_1 t + a_4, \qquad \xi^x = a_1 x + a_2 t + a_3, \qquad \eta^u = a_2, \qquad \eta^h = 0. \qquad (3)$$

These transformations provide the following three Lie point generators:

$$X_1 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad X_2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \qquad X_3 = \frac{\partial}{\partial x}$$

Consider the infinitesimal generators VA, VB defined by

$$V_A = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3$$
$$= \alpha_1 t\frac{\partial}{\partial t} + (\alpha_1 x + \alpha_2 t + \alpha_3)\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial u}$$
$$V_B = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$
$$= \beta_1 t\frac{\partial}{\partial t} + (\beta_1 x + \beta_2 t + \beta_3)\frac{\partial}{\partial x} + \beta_2\frac{\partial}{\partial u}$$

where $\alpha_i, \beta_i \in R$, such that $[VA\,;\,VB] = VAVB - VBVA = 0$, which yields $\alpha_1\beta_3 - \alpha_3\beta_1 = 0$. Since the system is invariant under the group generated by VA, we introduce a set of canonical variables defined by

$$V_A\bar{\tau} = 1, \qquad V_A\bar{\xi} = 0, \qquad V_A\bar{U} = 0, \qquad V_A\bar{P} = 0\,. \qquad (4)$$

allowing one to express VA as a translation with respect to $\bar{\tau}$, the characteristic conditions associated with (4) are

$$\frac{dt}{\alpha_1 t} = \frac{dx}{(\alpha_1 x + \alpha_2 t + \alpha_3)} = \frac{du}{\alpha_2} = \frac{d\bar{\tau}}{1}, \qquad (5)$$

where α_1, α_2 and α_3 are non-zero constants. Hence equation (5) yield the following transformation of variables

$$\bar{\tau} = \frac{1}{\alpha_1}\log t, \qquad \bar{\xi} = t^{-1}e^{\frac{(\alpha_1 x + \alpha_3)}{\alpha_2 t}}, \qquad \bar{U} = e^{u}\, t^{-\frac{\alpha_2}{\alpha_1}}, \qquad \bar{P} = h \qquad (6)$$

Now we can express V_B using the new variables as

$$\overline{V_B} = V_B\bar{\tau}\frac{\partial}{\partial\bar{\tau}} + V_B\bar{\xi}\frac{\partial}{\partial\bar{\xi}} + V_B\bar{U}\frac{\partial}{\partial\bar{U}} + V_B\bar{P}\frac{\partial}{\partial\bar{P}}$$
$$= \frac{\beta_1}{\alpha_1}\frac{\partial}{\partial\bar{\tau}} + \frac{(\alpha_1\beta_2 - \alpha_2\beta_1)}{\alpha_1}\bar{\xi}\frac{\partial}{\partial\bar{\xi}} + \frac{(\alpha_1\beta_2 - \alpha_2\beta_1)}{\alpha_1}\bar{U}\frac{\partial}{\partial\bar{U}}\frac{\partial}{\partial\bar{\xi}}$$

In a similar manner, we choose a second set of canonical variables allowing $\overline{V_B}$ to be written as translation with respect to ξ, i.e.

$$\overline{V_B}\tau = 0, \qquad \overline{V_B}\xi = 1, \qquad \overline{V_B}U = 0, \qquad \overline{V_B}P = 0. \qquad (7)$$

the characteristic conditions associated with (7) are

$$\frac{\alpha_1}{\beta_1}d\bar{\tau}=\frac{\alpha_1}{(\alpha_{1\beta_2-\alpha_2\beta_1)\bar{\xi}}}d\bar{\xi}=\frac{\alpha_1}{(\alpha_{1\beta_2-\alpha_2\beta_1)\bar{U}}}d\bar{U}=\frac{d\xi}{1}.$$

Solving the above characteristic equations, we obtain the following transformation of variables

$$\tau=\bar{\tau}-\log(\bar{\xi})^{\beta_1 K},\qquad \xi=\log(\bar{\xi})^{-\alpha_1 K},\qquad U=\bar{U}(\bar{\xi})^{\frac{-\alpha_2}{\alpha_1}},\qquad P=\bar{P}(\bar{\xi}). \tag{8}$$

where $K=\frac{1}{(\alpha_{1\beta_2-\alpha_2\beta_1})}$. Combining both (6) and (8), we are led to consider the following transformation of variables:

$$\tau=\log t^{\frac{\beta_1 K\alpha_2+1}{\alpha_1}}-\frac{\beta_1 K(x+\frac{\alpha_3}{\alpha_1})}{t},\qquad \xi=\log t^{-K\alpha_2}+\frac{\alpha_1 K(x+\frac{\alpha_3}{\alpha_1})}{t},\qquad u=\log U+\frac{(x+\frac{\alpha_3}{\alpha_1})}{t},\qquad p=P. \tag{9}$$

where U and P are arbitrary functions of τ and ξ. Using the above transformations in the governing system (1), we get the following system of PDEs

$$(\frac{\beta_1 K\alpha_2+1}{\alpha_1}-\beta_1 K\log U)\frac{\partial P}{\partial\tau}+(\alpha_1 K\log U-\alpha_2 K)\frac{\partial P}{\partial\xi}-\frac{\beta_1 KP}{U}\frac{\partial U}{\partial\tau}+\frac{\alpha_1 KP}{U}\frac{\partial U}{\partial\xi}+P=0 \tag{10}$$

$$(\frac{\beta_1 K\alpha_2+1}{\alpha_1}-\beta_1 K\log U)\frac{\partial U}{\partial\tau}+(\alpha_1 K\log U-\alpha_2 K)\frac{\partial U}{\partial\xi}-\frac{\beta_1 Kg(P+H)U}{P}\frac{\partial U}{\partial\tau}+\frac{\alpha_1 Kg(P+H)}{P}U\frac{\partial U}{\partial\xi}+U\log U=0 \tag{11}$$

By considering U = 1, the above system of PDEs can be reduced as follows

$$-\beta_1\frac{\partial P}{\partial\tau}+\alpha_1\frac{\partial P}{\partial\xi}=0,$$

$$\frac{\partial P}{\partial\xi}-\frac{\beta_1}{\alpha_1}\frac{\partial P}{\partial\tau}=0$$

(12)

Equation (12) can be solved as

$$P(\xi,\tau)=P_1(\eta) \tag{13}$$

Where $\eta=\tau-\frac{\beta_1}{\alpha_1}\xi$. Using (13) in the equation (11), we obtained

$$\frac{dP_1}{d\eta}+\alpha_1 P_1=0 \tag{14}$$

equation (14) can be solved

$$P_1=Ce^{-\eta\alpha_1} \tag{15}$$

where C is an arbitrary constant and thus, in view of the equations (9), (13) and (15) the solution of the system (1) can be expressed as follows

$$h=\frac{C}{t},\qquad u=\frac{(x+\frac{\alpha_3}{\alpha_1})}{t}.$$

(16)

It may be remarked that the state such as above, where the particle velocity exhibits linear dependence on the spatial coordinates, has been discussed by Clarke [13], Pert [14], and Sharma et al. [15]; Pert has shown that such a form of velocity distribution is useful in modelling the free fluids, and is attained in the large time limit.

3 . Conclusions

Lie group analysis is used to obtain special exact solutions of PDEs that describe one dimensional modified shallow water equations. This class of equations includes as a special case of the equations of classical shallow water equations. Lie symmetry groups, are used to transform the governing system of non-autonomous PDEs to autonomous system of PDEs. Exact solutions to the autonomous system of PDEs are obtained which in turn via the transformation of variables give special exact solutions to the governing system. Special exact solutions of a system of nonlinear PDEs are of great interest; these solutions play a major role in designing, analysing and testing of numerical methods for solving special initial-and/boundary value problems. The form of special solutions obtained here, such as the one where the particle velocity exhibits linear dependence on the special coordinate, has been discussed by some of the researchers in the past.

References

[1] G.W. Bluman, and , S. Kumei, *Symmetries and differential equations*, Springer, New York (1989).

[2] , L.V. Ovsiannikov, *Group analysis of differential equations*, New York: Academic, (1982).

[3] T. Raja Sekhar, and V.D. Sharma, *Similarity solutions for three dimensional Euler equations using Lie group analysis*, Appl. Math. Comput. 196(1), 147-157 (2008).

[4] M. Pandey, R. Radha and V.D. Sharma, *Symmetry analysis and exact solutions of magnetogasdynamic equations*, Quart. J. Mech. Appl. Math. 61(3), 291-310 (2008).

[5] Donato, A. and Oliveri, F., *Reduction to autonomous form by group analysis and exact solutions of axisymmetric MHD equations*, Math. Comput. Modelling 18(10), 83-90 (1993).

[6] F. Oliveri, and M.P. Speciale, *Exact solutions to the unsteady equations of perfect gases through Lie group analysis and substitution principles*, Internat. J. Non-Linear Mech. 37(2), 257-274 (2002).

[7] F. Oliveri, and M.P. Speciale, *Exact solutions to the ideal gasmagnetogasdynamics equations through Lie group analysis and substitution principles*, J. Phys. A 38(40), 8803-8820 (2005).

[8] W.F. Ames and A. Donato, *On the evolution of weak discontinuities in a state characterized by invariant solutions*, Internat. J. Non-Linear Mech. 23(2), 167-174 (1988).

[9] D. Sahin, N. Antar and T. Ozer, *Lie group analysis of gravity currents*, Nonlinear Anal. Real World Appl. 11(2) 167-174 (2010).

[10] E. Yasa, and T. Ozer,, *Conservation laws for one-layer shallow water wave systems*, Nonlinear Anal. Real World Appl. 11(2), 838-848 (2010).

[11] T. Raja Sekhar and V.D. Sharma, *Evolution of weak discontinuities in shallow water equations,* Appl. Math. Lett. 23(3) 327-330 (2010).

[12] T. Raja Sekhar and V.D. Sharma, *Interaction of shallow water waves*, Stud. Appl. Math. 121 (1) 1, 1-25(2008).

[13] J.F. Clarke, *Small amplitude gasdynamic disturbances in an exploding atmoshhere,* J. Fluid Mech. 89, 343-355(1978).

[14] G.J. Pert, *Self-similar flow with uniform velocity gradient and their use in modelling the free expansion of polytropic gases*, J. Fluid Mech. 100, 257-277 (1980).

[15] V.D. Sharma, R. Ram, and P.L. *Sachdev, Uniformly valid analytical solution to the problem of a decaying shock wave*, J. Fluid Mech. 185, 153-170 (1987).