2012 International Conference on Fluid Dynamics and Thermodynamics Technologies (FDTT 2012)
IPCSIT vol.33(2012)©(2012) IACSIT Press, Singapore

Free convection flow of Couple Stress fluid between parallel Disks

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Abstract. The steady incompressible couple stress fluid flow between parallel disks is considered. The lower and upper disks rotate with the same angular velocity Ω. Due to the rotating frame of reference, Coriolis and centrifugal forces appear explicitly in the momentum equation. The governing non-linear partial differential equations are transformed by a similarity transformation into a system of ordinary differential equations. The resulting equations are solved using Homotopy Analysis Method (HAM). The non-dimensional velocity and temperature profiles are displayed graphically for different values of the Prandtl number and couple stress fluid parameter.

Keywords: Natural convection, Couple stress fluid, HAM.

1. Introduction

Rotating-disk flow has long been an important topic in fluid dynamic research for the interests in practical as well as academic senses. Karman [1] introduced an ingenious similarity transformation to study the axisymmetric flow induced by a single rotating disk. Batchelor [2] showed that this transformation can be used even when the fluid is confined between two parallel disks rotating about a common axis at different speeds. Since then, several researchers considered the flow between two disks rotating about a common axis with the same angular velocity or different angular velocities. Recently, Batista [3] obtained analytical solution of the Navier–Stokes equations for the case of the steady flow of an incompressible fluid between two uniformly co-rotating disks. On the other hand, flow and Heat transfer associated with a rotating disk system has attracted the attention of researchers for the past few decades due to its numerous industrial and engineering applications of rotating machinery, lubrication, computer storage devices and crystal growth processes. Recently, Jiji and Ganatos [4] considered steady laminar flow and heat transfer generated by two infinite parallel disks.

Flow of non-Newtonian fluids between rotating disks has also drawn attention in view of its applications in engineering practice. Different models have been proposed to explain the behavior of non-Newtonian fluids. Among these, couple stress fluids introduced by Stokes [5] have distinct features, such as the presence of couple stresses, body couples and non-symmetric stress tensor. The main feature of couple stresses is to introduce a size dependent effect. Classical continuum mechanics neglects the size effect of material particles within the continua. This is consistent with ignoring the rotational interaction among particles, which results in symmetry of the force-stress tensor. The study of couple-stress fluids has applications in a number of processes that occur in industry such as the extrusion of polymer fluids, solidification of liquid crystals, cooling of metallic plate in a bath, and colloidal solutions etc. A review of couple stress (polar) fluid dynamics was reported by Stokes [6]. Recently, Srinivasacharya and Kaladhar [7] presented analytical solution of chemically reacting free convective couple stress fluid in an annulus with Soret and Dufour effects.

The homotopy analysis method [8] was first proposed by Liao in 1992, is one of the most efficient methods in solving different types of nonlinear equations such as coupled, decoupled, homogeneous and non- homogeneous. Also, HAM provides us a great freedom to choose different base functions to express solutions of a nonlinear problem [9]. The application of the Homotopy Analysis Method (HAM) in engineering problems is highly considered by scientists, because HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods. Recent developments of HAM, like convergence of HAM solution, Optimality of convergence control parameter discussed by Srinivasacharya and Kaladhar [10] for couple stress fluids.

In the present work the free convection in couple stress fluid with heat transfer flow between two infinite rotating disks is studied. The Homotopy Analysis method is employed to solve the governing nonlinear equations. Convergence of the derived series solution is analyzed. The behavior of emerging flow parameters on the velocity and temperature is discussed.

2. Formulation of the Problem

Consider the steady axisymmetric flow of a incompressible couple stress fluid between two parallel disks separated by a distance *d*. Choose the cylindrical coordinates (R, φ, Z) with the origin at the center of the lower disk. The lower disk $Z = 0$ is maintained at uniform constant temperature T_1 and concentration C_1, while the upper disk $Z = d$ at uniform constant temperature T_2 and concentration C_2. Assume that the lower and upper disks are rotating with the same angular velocity Ω. With respect to the rotating frame of reference, Coriolis and centrifugal forces appear explicitly in the momentum equation. The fluid has constant physical properties and the Boussinesq approximation is invoked for study of the centrifugal buoyancy effects. In the present study, the wall condition at lower disk is used as the reference state, at which the fluid confined by the disks lies at the temperature $T_r = T_1$ and rotates with the reference frame as a solid body, therefore, $U = V = W = 0$ and $\nabla P_r/\rho_r = \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{R} + g$. The velocity components in the R, φ and Z directions are U, V and W, respectively. Considering the laminar density temperature relation $\rho = \rho_r[1 - \beta(T - T_r)]$, the governing equations can be depicted as:

$$\nabla \cdot \bar{q} = 0 \tag{1}$$

$$(\bar{q}.\nabla)\bar{q} + 2\Omega_1[1-\beta(T-T_r)](\bar{e}_z \times \bar{q}) + \Omega_1^{\,2} R\beta(T-T_r)\bar{e}_r - g\beta(T-T_r)\bar{e}_z = -\nabla P^*/\rho_r - \nu\nabla\times\nabla\times\bar{q} - \eta_1/\rho_r\,\nabla\times\nabla\times\nabla\times\nabla\times\bar{q} \tag{2}$$

$$(\bar{q}.\nabla)T = \alpha\nabla^2 T \tag{3}$$

in which $P^* = P - P_r$ is the pressure departure from the reference condition, $\boldsymbol{\Omega} = \Omega_1 \bar{e}_z$, $\mathbf{R} = R\,\bar{e}_r$, and $\bar{e}_z$ and $\bar{e}_r$ are the unit vectors in the axial and radial directions, respectively. In the present analysis, the following dimensionless variables are used,

$$Z = \eta d, U = R\Omega F(\eta), V = R\Omega G(\eta), W = \sqrt{\nu\Omega}\,H(\eta), T - T_1 = \Delta T\,\theta$$

where $\Delta T = T_2 - T_1$ is the characteristic temperature difference. G, H, and θ are the tangential, axial velocities and the temperature function, respectively. The transformation is essentially of von Karman type but with an additional treatment to the temperature function in the energy equation. The governing equations (1)-(3) can thus be cast into the following dimensionless form:

$$S^2 H^{vi} - H^{iv} + 4\,\mathrm{Re}^{3/2}[(1+G)G' - B(G\theta' + G'\theta)] + \mathrm{Re}^{1/2} H H''' - 2B\,\mathrm{Re}^{3/2}\theta' = 0 \tag{4}$$

$$S^2 G^{iv} + \mathrm{Re}^{1/2}[HG' - H'G - H' + BH'\theta] - G'' = 0 \tag{5}$$

$$\theta'' - \mathrm{Pr}\,\mathrm{Re}^{1/2} H\theta' = 0 \tag{6}$$

in which the continuity equation

$$H' = 2\,\mathrm{Re}^{1/2} F \tag{7}$$

has been introduced to simplify the system by eliminating the radial velocity function $F(\eta)$.Here $Pr = \nu/\alpha$ is the Prandtl number,. $Re = d^2\Omega/\nu$ is the Reynolds number, $B = \beta\,\Delta T$ is the thermal Rossby number and

$S^2 = \eta_1/\mu d^2$ is the Couple stress parameter, The effects of couple-stress are significant for large values of S $(= l/d)$, where $l^2 = \eta_1/\mu$ is the material constant. The superscript (') denotes differentiation with respect to η, The boundary conditions are

$$H(0)=0, H'(0)=0, G(0)=0, H(1)=0, H'(1)=0, G(1)=0, \theta(0)=0, \theta(1)=1 \tag{8a}$$

$$H'''(0)=0, H'''(1)=0, G'(0)=0, G'(1)=0 \tag{8b}$$

The boundary condition (8b) implies that the couple stresses are zero at the Disc surfaces.

3. The HAM solution of the problem

For HAM solutions, we choose the initial approximations of $H(\eta)$, $H(\eta)$ and $\theta(\eta)$ as follows:

$$H_0(\eta)=0, G_0(\eta)=0, \theta_0(\eta)=\eta \tag{9}$$

and choose the auxiliary linear operators:

$$L_1 = \partial^6/\partial\eta^6,\ L_2 = \partial^4/\partial\eta^4 \text{ and } L_3 = \partial^2/\partial\eta^2 \tag{10}$$

such that

$$L_1(c_1+c_2\eta+c_3\eta^2+c_4\eta^3+c_5\eta^4+c_6\eta^5)=0,\ L_2(c_7+c_8\eta+c_9\eta^2+c_{10}\eta^3)=0,\ L_3(c_{11}+c_{12}\eta)=0 \tag{11}$$

where c_i (i = 1, 2, ..., 12) are constants. Introducing non-zero auxiliary parameters h_1, h_2 and h_3 we develop the zeroth-order deformation problems as follow:

$$(1-p)L_1\left[H(\lambda;p)-H_0(\lambda)\right] = ph_1N_1\left[H(\lambda;p)\right] \tag{12}$$

$$(1-p)L_2\left[G(\lambda;p)-G_0(\lambda)\right] = ph_2N_2\left[G(\lambda;p)\right] \tag{13}$$

$$(1-p)L_3\left[\theta(\lambda;p)-\theta_0(\lambda)\right] = ph_3N_3\left[\theta(\lambda;p)\right] \tag{14}$$

subject to the boundary conditions

$$\begin{aligned} &H(0;p)=H'(0;p)=G(0;p)=H'''(0;p)=G'(0;p)=0, \theta(0;p)=0 \\ &H(1;p)=H'(1;p)=G(1;p)=H'''(1;p)=G'(1;p)=0, \theta(1;p)=1 \end{aligned} \tag{15}$$

where $p \in [0, 1]$ is the embedding parameter and the non-linear operators N_1, N_2 and N_3 are defined as:

$$\begin{aligned} \mathrm{N}_1\left[\mathrm{H}(\eta;\mathrm{p}),\mathrm{G}(\eta;\mathrm{p}),\theta(\eta;\mathrm{p})\right] = S^2H^{vi} - H^{iv} + 4\,\mathrm{Re}^{3/2}[(1+G)G' - B(G\theta' + G'\theta)] + \mathrm{Re}^{1/2}H''' \\ -2B\,\mathrm{Re}^{3/2}\theta' \end{aligned} \tag{16}$$

$$\mathrm{N}_2\left[\mathrm{H}(\eta;\mathrm{p}),\mathrm{G}(\eta;\mathrm{p}),\theta(\eta;\mathrm{p})\right] = S^2G^{iv} + \mathrm{Re}^{1/2}[HG' - H'G - H' + BH'\theta] - G'' \tag{17}$$

$$\mathrm{N}_3\left[\mathrm{H}(\eta;\mathrm{p}),\mathrm{G}(\eta;\mathrm{p}),\theta(\eta;\mathrm{p})\right] = \theta'' - \mathrm{Pr}\,\mathrm{Re}^{1/2}H\theta' \tag{18}$$

For $p = 0$, we have the initial guess approximations

$$H(\eta;0)=H_0(\eta), G(\eta;0)=G_0(\eta), \theta(\eta;0)=\theta_0(\eta) \tag{19}$$

When $p = 1$, equations (12) - (14) are same as (4) - (6) respectively, therefore at $p = 1$ we get the final solutions

$$H(\eta;1)=H(\eta), G(\eta;1)=G(\eta), \theta(\eta;1)=\theta(\eta) \tag{20}$$

Hence the process of giving an increment to p from 0 to 1 is the process of $H(\eta; p)$ varying continuously from the initial guess $H_0(\eta)$ to the final solution $H(\eta)$ (similar for $G(\eta,p)$ and θ(η,p)). This kind of continuous variation is called deformation in topology so that we call system Eqs. (12) - (15), the zeroth-order deformation equation. Next, the m^th-order deformation equations follow as

$$L_1[H_m(\eta)-\chi_m H_{m-1}(\eta)] = h_1R_m^H, L_2[G_m(\eta)-\chi_m G_{m-1}(\eta)] = h_2R_m^G, L_3[\theta_m(\eta)-\chi_m\theta_{m-1}(\eta)] = h_3R_m^\theta \tag{21}$$

with the boundary conditions

$$\begin{aligned} &H_m(0)=H_m'(0)=G_m(0)=H_m'''(0)=G_m'(0)=H_m(1)=H_m'(1)=G_m(1)=H_m'''(1)=G_m'(1)=0 \\ &\theta_m(0)=\theta_m(1)=0 \end{aligned} \tag{22}$$

where

$$R_m^H(\eta) = S^2H^{vi} - H^{iv} + 4\,\mathrm{Re}^{3/2}\left[G' + \sum_{n=0}^{m-1}G_{m-1-n}'G_n - B\sum_{n=0}^{m-1}(G_n\theta_{m-1-n}' + G_{m-1-n}'\theta_n) - 0.5B\theta'\right] + \mathrm{Re}^{1/2}\sum_{n=0}^{m-1}H_nH_{m-1-n}''' \tag{23}$$

$$R_{\mathrm{m}}^{\mathrm{G}}(\eta) = S^2 G^{iv} + \mathrm{Re}^{1/2}\left[\sum_{n=0}^{m-1}(G'_{m-1-n}H_n - H'_{m-1-n}G_n + BH'_{m-1-n}\theta_n) - H'\right] - G'' \tag{24}$$

$$R_{\mathrm{m}}^{\theta}(\eta) = \theta'' - \Pr \mathrm{Re}^{1/2}\sum_{n=0}^{m-1} H_n \theta'_{m-1-n} \tag{25}$$

and, for m being integer $\chi_m = 0 \quad for\ m \le 1; \quad \chi_m = 1\ for\ m > 1$ (26)

The initial guess approximations $H_0(\eta)$),$G_0(\eta)$ and $\theta_0(\eta)$, the linear operators L_1, L_2 and L_3 and the auxiliary parameters h_1, h_2 and h_3 are assumed to be selected such that equations (12) - (15) have solution at each point $p \in [0, 1]$ and also with the help of Taylors series and due to eq. (19) $H(\eta; p)$,$G(\eta; p)$ and $\theta(\eta; p)$ can be expressed as

$$H(\eta;p) = H_0(\eta) + \sum_{m=1}^{\infty} H_m(\eta)p^m, G(\eta;p) = G_0(\eta) + \sum_{m=1}^{\infty} G_m(\eta)p^m, \theta(\eta;p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m \tag{27}$$

in which h_1, h_2 and h_3 are chosen in such a way that the series (27) are convergent at $p = 1$. Therefore we have from (20) that

$$H(\eta;p) = H_0(\eta) + \sum_{m=1}^{\infty} H_m(\eta), G(\eta;p) = G_0(\eta) + \sum_{m=1}^{\infty} G_m(\eta), \theta(\eta;p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) \tag{28}$$

for which we presume that the initial guesses to H, G and θ the auxiliary linear operators L and the non-zero auxiliary parameters h_1, h_2 and h_3 are so properly selected that the deformation $H(\eta; p)$, $G(\eta; p)$ and $\theta(\eta; p)$ are smooth enough and their mth -order derivatives with respect to p in equations (28) exist and are given respectively by $H_m(\eta) = \frac{1}{m!}\frac{\partial^m H(\eta;p)}{\partial p^m}\Big|_{p=0}$, $G_m(\eta) = \frac{1}{m!}\frac{\partial^m G(\eta;p)}{\partial p^m}\Big|_{p=0}$ $\theta_m(\eta) = \frac{1}{m!}\frac{\partial^m \theta(\eta;p)}{\partial p^m}\Big|_{p=0}$

It is clear that the convergence of Taylor series at $p = 1$ is a prior assumption, whose justification is provided via a theorem, so that the system in (28) holds true. The formulae in (28) provide us with a direct relationship between the initial guesses and the exact solutions. Moreover, a special emphasize should be placed here that the mth-order deformation system (21)-(24) is a linear differential equation system with the auxiliary linear operators L whose fundamental solution is known.

4. Results and Discussion

The expressions for H,G and θ contain the auxiliary parameters h_1, h_2 and h_3. As pointed out by Liao (2003), the convergence and the rate of approximation for the HAM solution strongly depend on the values of auxiliary parameter h. For this purpose, h-curves are plotted by choosing h_1, h_2 and h_3 in such a manner that the solutions (19)-(20) ensure convergence [9]. Here to see the admissible values of h_1, h_2 and h_3, the h-curves are plotted for 15th-order of approximation in Figs.(2)-(4) by taking the values of the parameters $Re = 50$, $Pr = 0.01$, $S = 1$ and $B = 0.05$.. It is clearly noted from Fig.2 that the range for the admissible values of h_1 is $-1.35 < h_1 < -0.6$. From Fig.3, it can be seen that the h-curve has a parallel line segment that corresponds to a region $-1.5 < h_2 < -0.6$. Fig.4 depicts that the admissible value of h_3 are $-1.25 < h_3 < -0.6$. A wide valid zone is evident in these figures ensuring convergence of the series.

The average residual errors are calculated at different order of approximations (m) and found that they are minimum at $h_1 = -1.03$, $h_2 = -1$ and $h_3 = -0.9$ respectively. Therefore, the optimum values of convergence control parameters are taken as $h_1 = -1.03$, $h_2 = -1$, $h_3 = -0.9$.

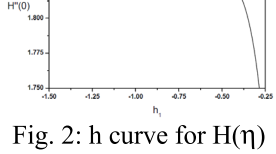

Fig. 2: h curve for H(η)

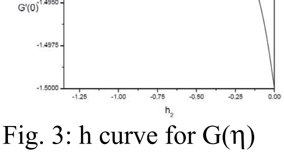

Fig. 3: h curve for G(η)

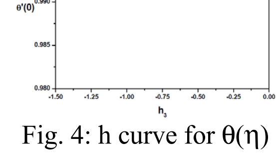

Fig. 4: h curve for θ(η)

The solutions for $F(\eta)$, $G(\eta)$, $H(\eta)$ and $\theta(\eta)$ have been computed and shown graphically in Figs. 5 to 12. The effects of couple stress fluid parameter (S) and Prandtl nuber have been discussed. To study the effects of S and Pr, computations were carried out by taking $B = 0.05$.

Figures 5 to 8 indicate the effect of the Couple stress fluid parameter S on *F(η),G(η), H(η)* and *θ(η)*. As the Couple stress fluid parameter S increases, the radial velocity *F(η*, the tangential velocity *G(η)* and axial velocity *H(η)* decrease. It is also clear that the temperature *θ(η)* decreases with an increase in *S*. It can be noted that the velocity in case of couple stress fluid is less than that of a Newtonian fluid case. Thus, the presence of couple stresses in the fluid decreases the velocity and temperature

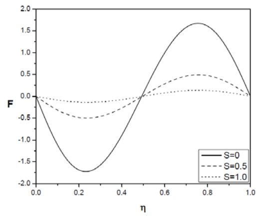

Fig. 5: Effect of *S* on *F* at *Pr* = 0.01

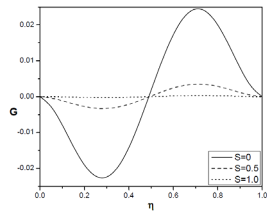

Fig. 6: Effect of *S* on *G* at *Pr* = 0.01

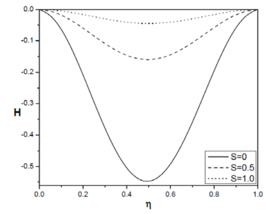

Fig. 7: Effect of *S* on *H* at *Pr* = 0.01

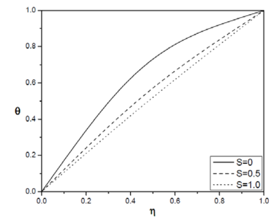

Fig. 8: Effect of *S* on θ at *Pr* = 0.01

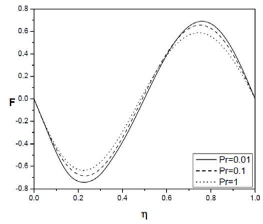

Fig. 9: Prandtl number *Pr* Effect on *F* at *S*=0.5

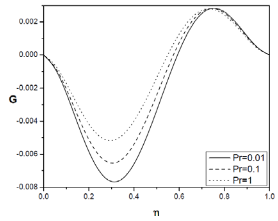

Fig. 10: Prandtl number *Pr* Effect on *F* at *S*=0.5

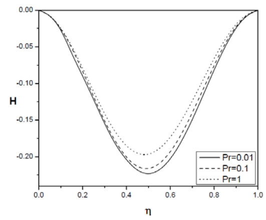

Fig. 11: Prandtl number *Pr* Effect on *F* at *S*=0.5

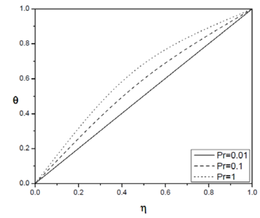

Fig. 12: Prandtl number *Pr* Effect on θ at *S*=0.5

In Figure 9 to 12 with B = 0.05, the Prandtl number effect significantly alters the flow fields. For large Prandtl number, *Pr* = 1, the temperature function changes abruptly in the thin thermal boundary layer but remain uniform in large portion of the wheel space. As *Pr* decreases from 1 to 0.01, the thermal diffusion is getting more and more important and, then, the temperature variation appears notably in the whole domain rather than confined in a narrow region. For small *Pr*, the temperature gradient near the disk 1, i.e. *Z* = 0, is alleviated. Due to coupling of the Coriolis induced buoyancy in circumferential fluid motion and the Prandtl number effects, salient *Pr*-dependence of the circumferential velocity is presented.

5. Conclusions

In this paper, the fully developed couple stress fluid flow between two parallel Discs has been studied. The governing equations are expressed in the non-dimensional form and are solved by using HAM. The features of flow characteristics are analyzed by plotting graphs and discussed in detail. The main findings are summarized as follows:

- The presence of couple stresses in the fluid decreases the velocity and temperature.
- Increase in the Prandtl number leads to the increase in temperature and decrease in velocity.

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