

Simultaneous Least Squares Wavelet Decomposition for Irregularly Spaced Data

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Abstract. Multiresolution analysis property of the Discrete Wavelet Transform (DWT) has become a powerful tool in signal reconstruction from regularly sampled data. For irregularly sampled data, however, other techniques should be used including the Least Square Wavelet Decomposition (LSWD) based on careful selection of wavelet basis. The commonly used method for LSWD is the level by level (sequential) wavelet decomposition which calculates the wavelet coefficients in each resolution separately. In this paper, we critically investigate this method and prove that it may result in gross interpolation error due to lack of discrete orthogonality for irregular data. We also show that its performance highly depends on the starting coarse resolution. We propose a different approach called the Simultaneous Least Square Wavelet Decomposition, which compute all wavelet coefficients simultaneously. We show the ability of the simultaneous approach to overcome the problem which leads to an excellent signal reconstruction capability.

Keywords: wavelet, multiresolution analysis, least squares, irregular data, signal reconstruction

1. Introduction

Recovery of a continuous function (or signal) from its discrete samples is one of the major activities in real world measurement. Traditional methods of signal reconstruction assume uniform spacing of the samples. However, actual data are not always uniformly sampled and may suffer from irregularities such as data dropout, multirate sampling, and sampling jitter. The problem of signal reconstruction from non-uniformly sampled data arises in diverse areas of science and engineering such as signal and image processing, computer graphics, communication systems, geo-science and identification and control.

The basic problem can be considered as the reconstruction of a multidimensional response surface from a limited number of irregularly spaced discrete samples, which are often noisy. From a mathematical view point, the fundamental problem is perhaps best considered as a function estimation problem. This allows us to use the substantial theoretical results and the ensuing algorithms developed for function estimation. A number of distinct approaches for the recovery of a function from irregular samples have been proposed.

In this study we are primarily concerned with the basis fitting methods, which approximate the function as the linear sum of a set of basic functions, in particular, the wavelet basis functions in a multiresolution structure[1],[2],[3]. Wavelets in all resolutions (or scale) are dilated and translated version of a single mother wavelet. They may also have very interesting Mathematical Properties such as orthogonality as well as concentrated energy in both time and frequency domain. For uniformly sampled data, the calculation of the wavelet coefficients can take advantage of the fast wavelet transform (FWT) algorithm [4]. In the case of non-uniformly sampled data, however, the FWT algorithm cannot be used directly and the estimation of the wavelet coefficients becomes more complex.

Many indirect and direct approaches have been suggested for handling irregular data. A useful indirect method involves the projection of the irregular data onto a regular grid using a suitable projection technique [5]. Once the data has been projected, we are free to take full advantage of the well-developed machinery of the FWT algorithm with its attendant computational efficiency. A serious drawback of this approach lays in the difficulty of its extension to two and higher dimensional function estimation problems.

In direct methods, however, the available irregular data is used directly without any manipulation. Two popular schemes for direct handling of irregular data within the multiresolution wavelet structure are : the

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lifting scheme and the least square approach. The Lifting Scheme popularized by Wim Sweldens and his co-workers [6],[7] to employ *data adapted* basis functions or the so called *second generation wavelets*. The chief advantage of the lifting scheme lies in its ability to provide the perfect reconstruction of the original irregular data, which is important in some application such as computer graphics. Perfect reconstruction is not, however, a primary aim in denoising applications. It has inherent shortcoming for multidimensional function estimation from noisy data sets.

Another direct method is the least square wavelet approach which is favored for denoising applications in high dimensions [8],[9],[10]. See also [11] for Multiresolution basis fitting for irregular data using scaling functions alone. In the least squares approach, the basis functions are in fact the same as the basis functions used in the regular setting and hence the translated and dilated versions of the same mother wavelet. In contrast to the projection and lifting approaches, however, we do not necessarily use the entire set of basis functions defined within the multiresolution structure. Instead at each resolution we first identify and *select* those basis functions that can contribute significantly to the approximation. We then set up and solve an optimization problem (least squares) to obtain the coefficients for the *selected* basis functions. In Section 2, we investigate the least squares wavelet decomposition approach starting from selection of basis functions and proceed to formulations of both level by level and simultaneous LSWD. Results and discussion will come in Section 3 to prove the gross interpolation error of commonly used sequential LSWD and the excellent performance of the proposed simultaneous LSWD. Section 4 give a brief conclusions.

2. The Least Squares Wavelet Decomposition

The general approach described in this section relies on the formulation and solution of a variety of the least squares problems to determine the coefficients in the multiresolution wavelet expansion,

$$F_L(x) = \sum_{k=1}^{N_J} c_{J,k} \varphi_{J,k}(x) + \sum_{j=J}^{L-1} \sum_{k=1}^{N_j} d_{j,k} \psi_{j,k}(x) \quad (1)$$

The least squares wavelet approach, which has also been employed by a number of other authors (Bakshi and Stephanopoulos 1993 [8], Safavi, 1995 [9], Ford and Etter, 1998 [10]) and so on, is critically reviewed in this section.

In the case of regular data, the basis functions are defined on a multiresolution grid with nodes that precisely coincide with the data points. Each basis function is therefore seen by at least two data points and all basis functions make a significant contribution and are included in the expansion of the estimated function. Indeed in the FWT algorithm it is not necessary to consider the basis functions explicitly and it is sufficient to deal with the corresponding filter coefficients. The situation is substantially more complex with irregular data. This is because the spacing of the data does not coincide with the spacing of the regular multiresolution grid. There is in fact no longer a guarantee that each wavelet basis function has at least two data points in its (effective) compact support.

Adding a basis function which is not seen by the data to the multiresolution expansion does not help in finding a closer approximation and may result in severe instabilities. When data is non-uniformly distributed in the input domain, there may be some regions which are denser than others. The dense regions can support a higher resolution than the sparse ones. This is because as we go up in resolution the basis functions become narrower and the probability of being seen by sparse data reduces.

2.1 Selection of Basis Functions

The application of the least squares method calls for identifying and putting aside those (regular) basis functions that cannot contribute to the wavelet expansion. The key criterion for including a basis function in the expansion is whether it contains *sufficient* data in its support. Various procedures can be developed by associating a particular wavelet to each tile in the space-frequency (or space-resolution) tiling of the wavelet system, the natural choice is the wavelet that is centered on each tile. The multiresolution tiling and the centre of the wavelets are shown on Figure 4.14 with a particular irregular data set superimposed. The key question to answer is whether there is sufficient data in a tile so that we may select its associated wavelet. It is clear that wavelets associated with tiles containing less than two data points should not be selected, otherwise the number of total basis functions at all resolution becomes more than the number of data points .

In addition, the principal function of a multiresolution structure is to capture the difference between (at least two) data values by going to progressively higher resolutions with dilated wavelets. Included on Figure 1, are the locations of a set of arbitrarily spaced data points. The tiles containing less than two data points are shown as crossed tiles on Figure 1. We note immediately that with irregular data, different regions can support different resolutions based on the *local* density of the data.

We start with the simplest Haar wavelet system, which is sketched in Figure 2 at two successive resolutions. In order to select a Haar wavelet there must be at least one data point within each half of its support, it is only then that we can characterize the difference between these points. A tile with both data points on one half of the support is not chosen and is shown as a crossed tile on Figure 2. We see from the left hand panel on figure 2 that the selection at each resolution can be performed independently.

2.2 Computing the Wavelet Coefficients: Formulating the Least Squares Decomposition

Once the set of basis functions has been selected, it remains to form and solve a least squares problem to determine the coefficients in the multiresolution wavelet expansion. Consider the multiresolution wavelet expansion of a function written in its expanded form,

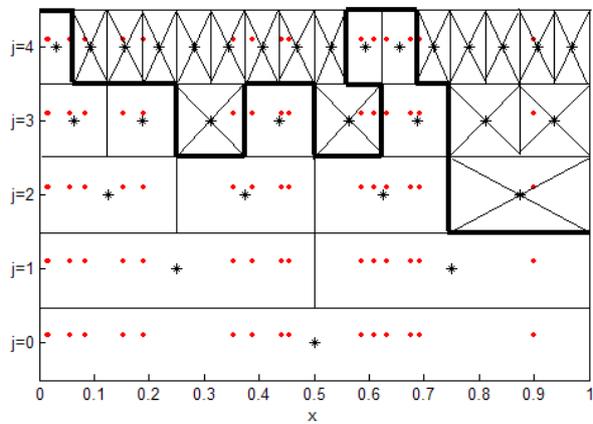


Figure 1. The spatial variation in the local resolution supported by a typical irregular data set. Tiles with less than two data points are crossed.

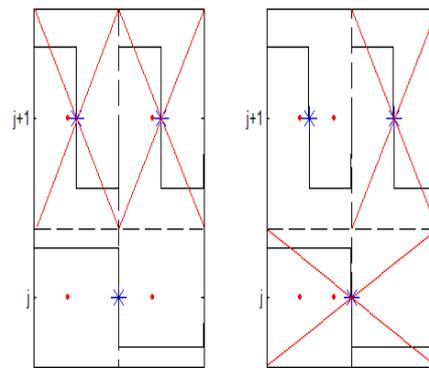


Figure 2. Selection of Haar wavelets based on the location of data, (*) centre of wavelet, and (.) location of data. Crossed tiles are not selected.

$$F_L(x) = \sum_{k=1}^{N_J} c_{J,k} \varphi_{J,k}(x) + \sum_{k=1}^{N_J} d_{J,k} \psi_{J,k}(x) + \sum_{k=1}^{N_{J+1}} d_{J+1,k} \psi_{J+1,k}(x) \dots + \sum_{k=1}^{N_{L-1}} d_{L-1,k} \psi_{L-1,k}(x) \quad (2)$$

Here $\varphi_{j,k}(x)$ and $\psi_{j,k}(x)$ are the dilated and translated versions of a single scaling function and a single mother wavelet respectively. The coarsest resolution is indicated by J and N_J is the number of basis functions selected at the coarsest resolution. Similarly, N_j represents the number of wavelet basis functions selected at resolution $j > J$ and L represents the maximum finest resolution considered. Given M irregularly spaced samples of an unknown function in the interval $I=[0, 1]$, $(x_i, y_i), i = 1, \dots, M$, our objective is to determine the coefficients in the above expansion so that the function may be evaluated at any arbitrary location within the interval I .

The coefficients $c_{j,k}$ and $d_{j,k}$ can be determined by forming and solving least squares problems. The procedure used to formulate the least squares problem can seriously affect the quality of the reconstruction, a point that has been largely overlooked. In the reported applications of the least squares wavelet decomposition technique [8],[9],[10], the coefficients are determined in a level by level (sequential) procedure. The level by level approach has a number of inherent shortcomings; in particular it may lead to gross errors in reconstruction. As an alternative we develop a formulation in which all the coefficients are determined simultaneously. The simultaneous formulation, which to our knowledge has not been previously reported, overcomes the difficulty associated with the level by level sequential decomposition and can

provide better reconstruction with irregular data. We shall first present the sequential and simultaneous formulation of the least squares decomposition procedures and follow this by a careful assessment of their characteristic performance in section 3.

2.3. Level by Level (Sequential) Approach

In the level by level (sequential) approach, we start from the coarsest resolution J and as a first attempt try to approximate the unknown function with scaling functions alone,

$$F_J(x) = \sum_{k=1}^{N_J} c_{J,k} \varphi_{J,k}(x) \quad (2)$$

The coefficients $c_{J,k}$ are found by solving the following least squares problem

$$\min_{\underline{c}_J} \mathfrak{R}_{emp}(\underline{c}_J) = \frac{1}{2} (\underline{y} - G_J \underline{c}_J)^T (\underline{y} - G_J \underline{c}_J) \quad (3)$$

where $\underline{c}_J = (c_{J,1}, \dots, c_{J,N_J})^T$ is the vector of coefficients and the (design) matrix G_J is given by

$$G_J = \begin{bmatrix} \varphi_{J,1}(x_1) & \cdots & \varphi_{J,N_J}(x_1) \\ \vdots & \ddots & \vdots \\ \varphi_{J,1}(x_M) & \cdots & \varphi_{J,N_J}(x_M) \end{bmatrix} \quad (4)$$

The minimization of $\mathfrak{R}_{emp}(\underline{c}_J)$ reduces to the solution of the linear system

$$(G_J^T G_J) \underline{c}_J = G_J^T \underline{y} \quad (5)$$

The first estimation of the function at the data points is given by $\hat{\underline{y}}_J = G_J \underline{c}_J$ (6)

and its residual with the measured values is defined as $\underline{r}_J = \underline{y} - \hat{\underline{y}}_J$ (7)

We can view the residuals \underline{r}_J as the detail not captured by the scaling functions and attempt to fit them with the wavelets selected at resolution J , by minimizing

$$\min_{\underline{d}_J} \mathfrak{R}_{emp}(\underline{d}_J) = \frac{1}{2} (\underline{r}_J - H_J \underline{d}_J)^T (\underline{r}_J - H_J \underline{d}_J) \quad (8)$$

where $\underline{d}_J = (d_{J,1}, \dots, d_{J,N_J})^T$ is the vector of detail coefficients and the matrix H_J is given by

$$H_J = \begin{bmatrix} \psi_{J,1}(x_1) & \cdots & \psi_{J,N_J}(x_1) \\ \vdots & \ddots & \vdots \\ \psi_{J,1}(x_M) & \cdots & \psi_{J,N_J}(x_M) \end{bmatrix} \quad (9)$$

This minimization in turn reduces to the solution of the linear system

$$(H_J^T H_J) \underline{d}_J = H_J^T \underline{r}_J \quad (10)$$

and the next approximation to the measured values is given by

$$\hat{\underline{y}}_{J+1} = \hat{\underline{y}}_J + \hat{\underline{r}}_J = (G_J \underline{c}_J + H_J \underline{d}_J) \quad (11)$$

The detail not captured so far is given by the updated residuals $\underline{r}_{J+1} = \underline{y} - \hat{\underline{y}}_{J+1}$ (12)

and an attempt is made to capture this detail with the wavelets selected at resolution $J+1$,

$$\min_{\underline{d}_{J+1}} \mathfrak{R}_{emp}(\underline{d}_{J+1}) = \frac{1}{2} (\underline{r}_{J+1} - H_{J+1} \underline{d}_{J+1})^T (\underline{r}_{J+1} - H_{J+1} \underline{d}_{J+1}) \quad (13)$$

Solving the associated linear system, $(H_{J+1}^T H_{J+1}) \underline{d}_{J+1} = H_{J+1}^T \underline{r}_{J+1}$ (14)

provides the coefficients \underline{d}_{J+1} and the next approximation is given by

$$\hat{\underline{y}}_{J+2} = \hat{\underline{y}}_{J+1} + \hat{\underline{r}}_{J+1} = (G_J \underline{c}_J + H_J \underline{d}_J) + H_{J+1} \underline{d}_{J+1} \quad (15)$$

Repeating this process up to the finest resolution L provides all the coefficients $c_{j,k}$ and $d_{j,k}$ required for the multiresolution approximation.

2.4. Simultaneous Decomposition

As an alternative to the level by level decomposition, we may attempt to recover all the scaling and wavelet coefficients simultaneously. This is achieved by appending all the coefficient vectors together to form a single vector:

$$\underline{a}^T = (\underline{c}_J^T | \underline{d}_J^T | \underline{d}_{J+1}^T | \cdots | \underline{d}_{L-2}^T | \underline{d}_{L-1}^T), \quad (16)$$

forming a large design matrix

$$A = [G_J | H_J | H_{J+1} | \cdots | H_{L-2} | H_{L-1}] \quad (17)$$

$$\text{The overall linear system, } A\underline{a} = \underline{y} \quad (18)$$

is in general over-determined and does not admit a solution. A solution which minimizes the sum of squared errors is obtained by attempting to minimize

$$\min_{\underline{a}} \mathfrak{R}_{emp}(\underline{a}) = \frac{1}{2}(\underline{y} - A\underline{a})^T(\underline{y} - A\underline{a}) \quad (19)$$

The solution to this large minimization problem is obtained by solving the linear system,

$$(A^T A)\underline{a} = A^T \underline{y} \quad (20)$$

The level by level and simultaneous formulation of the least squares wavelet decomposition yield identical results for regularly spaced data. In such cases discrete orthogonality ensures that the design matrices at different resolutions are mutually orthogonal,

$$\begin{aligned} G_J^T H_j &= [0] \quad J \leq j < L \\ H_j^T H_k &= [0] \quad J \leq j \text{ and } k < L, j \neq k \end{aligned} \quad (21)$$

and the overall system reduces to:

$$\begin{bmatrix} G_J^T G_J & [0] & \cdots & [0] \\ [0] & H_J^T H_J & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots \\ [0] & [0] & \cdots & H_{L-1}^T H_{L-1} \end{bmatrix} \begin{bmatrix} \underline{c}_J \\ \underline{d}_J \\ \vdots \\ \underline{d}_{L-1} \end{bmatrix} = \begin{bmatrix} G_J^T \underline{y} \\ H_J^T \underline{y} \\ \vdots \\ H_{L-1}^T \underline{y} \end{bmatrix} \quad (22)$$

We can also express all but the first element of the right hand side vector in (22) in terms of successive residuals. For example, noting that $\tilde{\underline{y}}_J = G_J \underline{c}_J$ and $\underline{r}_J = \underline{y} - \tilde{\underline{y}}_J$, we have

$$H_J^T \underline{r}_J = H_J^T(\underline{y} - \tilde{\underline{y}}_J) = H_J^T \underline{y} - [H_J^T G_J] \underline{c}_J = H_J^T \underline{y} \quad (23)$$

For regular data, therefore, discrete orthogonality ensures that the overall system (4.52) is reduced to exactly the same decoupled sequence of smaller least squares problems solved in the level by level method. Discrete orthogonality does not hold for irregularly spaced data and the level by level design matrices are no longer strictly orthogonal. Different results can therefore be expected from the applications of level by level and simultaneous least squares decompositions. The application of the level by level least squares wavelet decomposition for irregular data can lead to gross interpolation error due to lack of discrete orthogonality. On the other hand, the sequential determination of the wavelet coefficients from successively smaller residuals imparts a degree of stability against improper wavelet selection. The simultaneous least squares decomposition does not rely on discrete orthogonality and therefore does not suffer from gross interpolation error. However, it is more sensitive to improper wavelet selection because it operates directly on the data vector rather than the residuals.

3. Results and Discussion

In the case of regular data, the data spacing (the finest resolution) is not spatially variable and a single finest resolution applies globally. This offers an opportunity to develop the highly efficient transformation embodied in the FWT algorithm. In the case of irregular data, the finest resolution is spatially variable and is dependent on the local density of the data. In this section we explore the performance of the level by level (sequential) and the alternative simultaneous least squares wavelet decompositions for simple but carefully chosen irregularly spaced examples. Our objective is to highlight clearly the advantages and shortcoming of each technique and, where possible, make suggestions for improving the performance.

3.1 Gross Interpolation Error in the Level by Level Least Squares Wavelet Decomposition for Irregularly Spaced Data

In the case of irregularly spaced data, the reconstructed function obtained by the level by level least squares decomposition may exhibit large deviation from the original data points. This gross interpolation error is strongly influenced by the starting coarse resolution selected. This point has not been directly addressed in previous literature, most probably because the irregular data sets employed did not deviate significantly from regular spacing. It is particularly surprising that this problem has not been discussed in articles on wavelet neural networks (Wave-Net), which are aimed specifically at handling irregular data (Bakshi and Stephanopoulos 1993[8], Safavi, 1995 [9]). To highlight the serious nature of this difficulty we consider a carefully constructed irregular data set and the simplest Haar wavelet system. Figure 3 shows the level by level reconstruction of a sine wave from 50 irregular samples starting from 4 different coarse resolution at $J=0, 2, 4,$ and 6 . The data set consists of $M=50$ discrete samples of a sine wave located at

$$x_i = \left(\frac{i}{M}\right)^2, \quad i = 0, 1, \dots, M - 1 \quad (24)$$

Selecting the sample locations in this way strongly emphasizes the irregularity of the data at low x values. It is striking that the starting coarse resolution has such a strong influence on the form and quality of the reconstructed function. Starting from the coarsest possible resolution $J=0$ gives a reconstruction that is strongly biased and deviates significantly from the sample values over the entire interval. Setting the coarse resolution at $J=2$, the major deviation from sample values is confined to $x < 0.25$, where the irregularity is stronger. Starting from an even higher coarse resolution $J=4$ gives a better approximation and the bias is confined to $x < .0625$, where the irregularity is strongest. An overall measure of deviation from the sample values is the sum of squared error (SSE), which is 2.38 at $J=0$, 0.86 at $J=2$ and 0.043 at $J=4$.

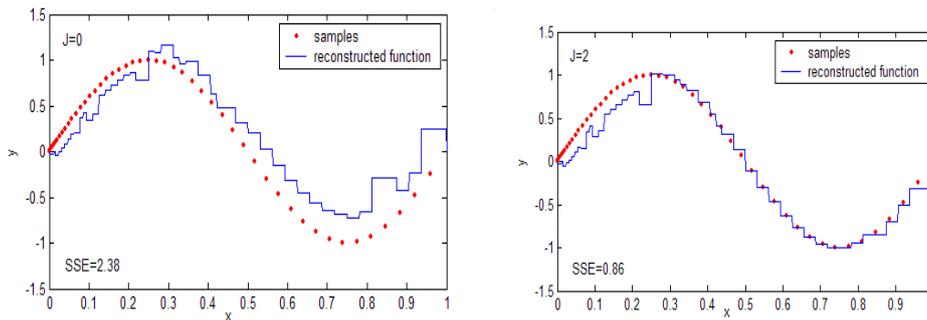


Figure 3 Influence of the starting coarse resolution on the performance of the level by level least squares wavelet decomposition

Noting that gross reconstruction error does not arise with regularly spaced dyadic data, but can be severe for irregularly spaced data. Figure 4 compares the reconstruction of a sine wave from 64 regular samples with that from 64 irregular samples. In both cases the level by level least squares decomposition was started at the coarsest resolution $J=0$. The data is strictly interpolated for the regular case but shows gross trend error in the irregular case.

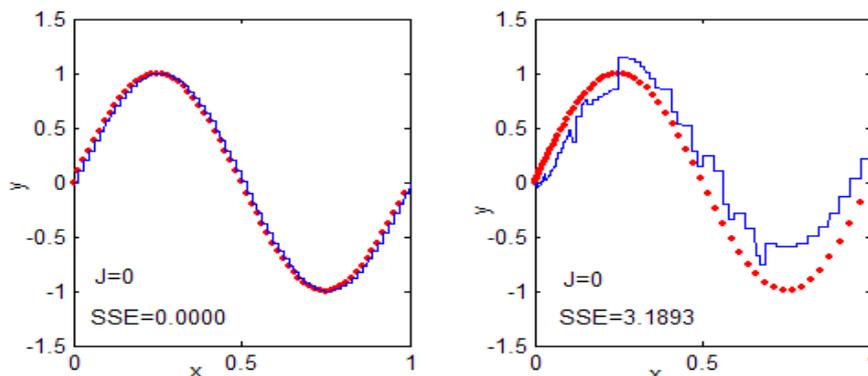


Figure 4 Reconstruction of a sine wave from 64 regular and 64 irregular samples using the level by level least squares decomposition with $J=0$

The fundamental reason of the large difference between the reconstruction qualities is the absence of discrete orthogonality in the case of irregular data. In a wavelet multiresolution structure, the approximation at level $j+1$ is built up by adding the detail at level j to the approximation at level j :

$$F_{j+1}(x) = F_j(x) + f_j(x) \quad (25)$$

The discrete analog of the above relation in terms of the wavelet coefficients is given by:

$$G_{j+1}c_{j+1} = G_j c_j + H_j d_j \quad (26)$$

In the case of dyadic regularly spaced data, the sub-spaces of the scaling functions and the wavelets are designed to be the orthogonal complement of each other. This ensures that discrete orthogonality holds and the discrete refinement relation (26) is satisfied exactly. The starting coarse resolution J in the level by level least squares decomposition is then immaterial and the same reconstruction is obtained starting from any chosen coarse resolution. For irregularly spaced data, however, discrete orthogonality is not guaranteed and the coefficients computed by the level by level decomposition do not satisfy the discrete refinement relation (26) precisely. Consequently, the choice of the starting coarse resolution can strongly affect the quality of the reconstruction from irregular data as demonstrated on Figure 4.

The improvement in reconstruction quality observed in Figure 3 on increasing the starting coarse resolution J can be clearly explained for the Haar wavelet system employed. The task of the Haar scaling function is to capture the local average of the signal. The Haar wavelets, in common with all other wavelets, have a zero mean and their linear combination cannot alter the local averages. In the regular case, the sample averages coincide with the averages of the sampled function in each tile of the multiresolution structure. The Haar scaling functions at each resolution can therefore capture the local signal average closely within each tile. We can therefore start the level by level reconstruction at any coarse resolution and the reconstructed signal will interpolate the regularly sampled data exactly.

For irregularly spaced data, however, the sample average within any given tile may differ markedly from that of the true function. The Haar scaling functions will then capture the (incorrect) sample average rather than the true signal average. The Haar wavelets added at the higher resolutions cannot alter such incorrect averages and this may lead to gross reconstruction error. The advantage of starting at a higher coarse resolution is due to the larger number of narrower scaling functions employed. Reducing the support of the scaling functions tends to reduce the effect of sample irregularity within the associated tiles.

The above arguments were confined to the simple Haar wavelet system for reasons of clarity. The same arguments, however, apply to any other wavelet system provided we consider weighted averages rather than simple averages. To reduce the potential for gross interpolation error for irregularly spaced data, the starting coarse resolution in the level by level least squares decomposition must be carefully selected. There appears to be no problem independent procedure available for such a selection and in our opinion a better approach is to resort to the simultaneous least squares decomposition, with all the coefficients determined at once rather than sequentially. See [12] for more details.

3.2 Excellent Approximation Capability of the Simultaneous Least Squares Wavelet Decomposition

In the simultaneous least squares decomposition all of the wavelet coefficients and the scaling coefficients are determined together by forming and solving a single linear system (20). Furthermore, no reference is made to the discrete refinement relation (26) in formulating this linear system. This allows the wavelet coefficients to adapt so that the details at the various resolutions are captured simultaneously and enables the scaling coefficients at the chosen coarse resolution to capture the local average of the signal accurately.

Figure 5 shows the reconstruction obtained from 64 regular and 64 irregular samples of a sine wave using the simultaneous least squares decomposition with the Haar wavelet system. It is evident that the irregular samples are interpolated perfectly with an $SSE=3.6 \times 10^{-31}$. Using the level by level decomposition on the same irregular data set exhibited gross reconstruction error with an $SSE=3.19$, see Figure 4.

The perfect reconstruction of the irregular data observed in Figure 5 is a special feature of the Haar wavelet system. For the simple Haar system, the number of wavelets selected N equals the number of the

irregular data points $M=N$. This turns the design matrix A in system (18) into a full rank $N \times N$ matrix permitting perfect interpolation. For other wavelets with more complex shapes, some wavelets may contain insufficient data in their (effective) compact support and will not be selected. The number of selected basis functions is then smaller than the number of the irregular data points, $N < M$, making the linear system (18) over-determined. The least squares solution obtained by minimizing the empirical risk (19) will no longer achieve perfect interpolation. Figure 6 compares the reconstruction obtained from 64 regular and irregular samples of a sine wave using the cubic B-spline wavelet systems. In the irregular case, perfect interpolation is not achieved but the irregular data is still well interpolated with an $SSE=1.7 \times 10^{-3}$.

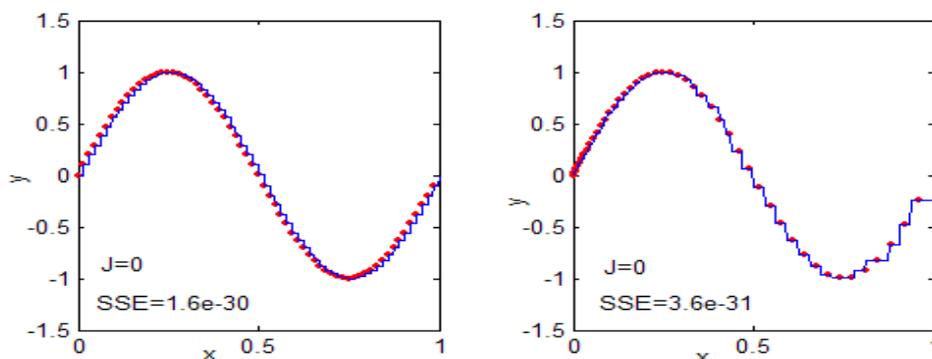


Figure 5 Reconstruction of a sine wave from 64 regular and 64 irregular samples using the simultaneous least squares decomposition with $J=0$

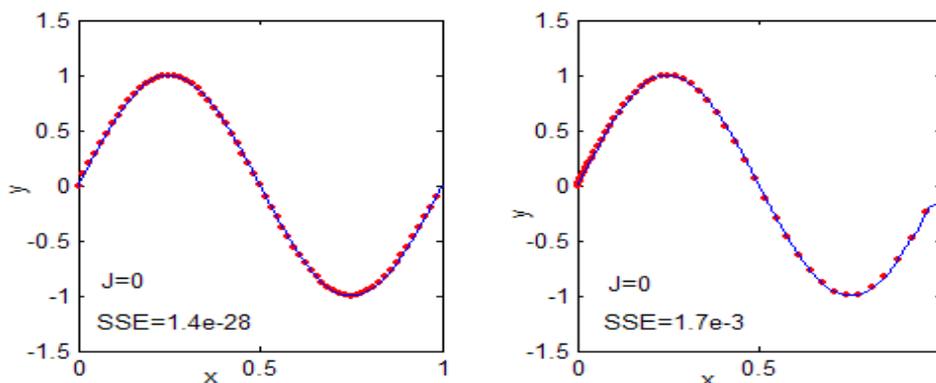


Figure 6 Reconstruction of a sine wave from 64 regular and 64 irregular samples using the simultaneous least squares decomposition using cubic B-spline wavelets

4. Conclusions

We proved that the commonly used least square wavelet decomposition (LSWD) may suffer from gross interpolation error in the case of strong irregularity. This error may be reduced by proper selection of the starting coarse resolution. To overcome this problem, we proposed a simultaneous approach which compute the scaling and wavelet coefficients at all resolution together. We showed that it has perfect reconstruction property in the case of Haar wavelet and the excellent result for the other continuous wavelet. The authors have also extended the simultaneous method to the multidimensional case and examined this superiority. The simultaneous approach, however, is more sensitive to wavelet selection in the finest resolution. The authors have concentrated on a hybrid approach, which combine the improved reconstruction property of the simultaneous method and the stability of the level by level method.

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