

Optimal Control of Discrete-Time Linear Stochastic Dynamic System with Model-Reality Differences

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Abstract. A rapid development of digital computer has brought great changes to the control systems. The control schemes are altered from optimizing the continuous-time dynamic system to discretization design of continuous plants. Also, there are many processes in discrete nature and only can be solved by discrete time controller. In this paper, a discussion of optimal control of discrete-time linear stochastic dynamic system is made. We modify the novel optimal control algorithm which developed by Roberts [1] and Becerra [2] to obtain the solution of linear stochastic dynamic system in spite of model-reality differences. This extension principle of model-reality differences is taking into account the state estimation by Kalman filtering, and integrating the system optimization and parameter estimation to give the optimum of real plant. During the iterative computations, the model-based optimal control problem is solving in which the real optimum is obtained after the convergence is achieved. A scalar example is presented and the resulting graphical solutions show that the proposed algorithm is efficient to provide the optimum of real plant.

Keywords: stochastic linear optimal control, principle of model-reality differences

1. Introduction

Optimization and control of stochastic system are challenge work. Since finding the optimal decision to minimize the cost functional subject to a set of dynamic state that disturbed by random noises is more difficult rather than deterministic situation. We can measure the entire state and suggest the admissible of decision ideally under linear quadratic regulator (LQR) approximation optimization. However, in presence of random disturbances, our decision is not necessary appropriate.

The principle of model-reality differences approach, known as DISOPE (dynamic integrated system optimization and parameter estimation), leads the nonlinear optimal control problems to be solved without random disturbances. It takes into account the different structural and parameters between the real plant and the model used in its iterative computations. The repeated solution generated by system optimization and parameter estimation converges to the correct real optimum in spite of the differences among the model used and real plant. In this manner, we only solve the modified model-based optimal control problem in order to obtain the optimum of real plant [1], [2].

Integrate the system optimization and parameter estimation with consideration of the random environment inspires us to investigate the optimization and control stochastic system. In this paper, we analyze a modification of the principle of model-reality differences and propose the extension principle to overcome the problems of stochastic control in spite of differences between the estimated real plant and the model used.

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Towards this goal, this paper is organized in the following manner. In the Section 2, we describe the general discrete-time stochastic optimal control problem and the model-based optimal control problem. In Section 3, we analyze an extension principle of model-reality differences and in Section 4, the solutions of modified model-based optimal control problem are summarized. In Section 5, a scalar example is presented and the graphical solutions show that the proposed algorithm is efficient to provide the optimum of linear stochastic optimal control problem.

2. Problem Statement

Consider the following stochastic optimal control problem (SOP).

$$\text{Minimize } J_S^* = \varphi^*(x(N)) + \sum_{k=0}^{N-1} [L^*(x(k), u(k), \omega(k))] \quad (1)$$

$$\text{subject to the } x(k+1) = f^*(x(k), u(k), \omega(k)) \text{ and } y(k) = h^*(x(k)) + \eta(k),$$

where $u(k) \in \mathfrak{R}^m$, $x(k) \in \mathfrak{R}^n$ and $y(k) \in \mathfrak{R}^p$ are the discrete control, random state sequences and measured output, respectively, $\varphi^*(\cdot): \mathfrak{R}^n \rightarrow \mathfrak{R}$ is a real terminal measure, $L^*(\cdot): \mathfrak{R}^n \times \mathfrak{R}^m \times \mathfrak{R}^n \rightarrow \mathfrak{R}$ is the real performance measure function, $f^*(\cdot): \mathfrak{R}^n \times \mathfrak{R}^m \times \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ represents the stochastic reality, $h^*(\cdot): \mathfrak{R}^n \rightarrow \mathfrak{R}^p$ is the real measured output function, $\omega(k) \in \mathfrak{R}^q$ is the process noise that represents disturbances or modeling inaccuracies and $\eta(k) \in \mathfrak{R}^p$ is measurement noise that due to sensor inaccuracy. The random state, process noise and measurement noise are uncorrelated and their probability density distributions are given as follow:

$$\left. \begin{aligned} x(0) &\sim \mathcal{N}(\bar{x}_0, M_0), \quad E(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T = M(0) \\ \omega(k) &\sim \mathcal{N}(\mathbf{0}, Q_\omega), \quad E\omega(k)\omega(k)^T = Q_\omega \\ \eta(k) &\sim \mathcal{N}(\mathbf{0}, R_\eta), \quad E\eta(k)\eta(k)^T = R_\eta \end{aligned} \right\} \quad (2)$$

where \bar{x}_0 is mean of initial state. The covariance matrices are defined as $n \times n$ state covariance, $M(0)$, $q \times q$ process covariance, Q_ω , and $p \times p$ measurement covariance, R_η , which are known *a priori*. Obviously, the distributions are known as zero-mean Gaussian white noise distribution.

The stochastic Hamiltonian function [3] is defined as follow:

$$H^*(\cdot) = L^*(x(k), u(k)) + p(k+1)^T f^*(x(k), u(k), \omega(k)) \quad (3)$$

and the necessary optimality conditions evaluated at $(x(k), u(k)) = (\hat{x}(k), u(k))$ are

$$\left. \begin{aligned} \nabla_{u(k)} H(\cdot) &= \mathbf{0} \\ \nabla_{x(k)} H(\cdot) - p(k) &= \mathbf{0} \\ \nabla_{p(k+1)} H(\cdot) - x(k+1) &= \mathbf{0} \end{aligned} \right\} \quad (4)$$

Instead of solving the SOP, the following linear estimated stochastic model-based optimal control problem (MOP) is considered.

$$\text{Minimize } J_M = \frac{1}{2} \bar{x}(N)^T S(N) \bar{x}(N) + \gamma_1 + \frac{1}{2} \sum_{k=0}^{N-1} [\bar{x}(k)^T Q \bar{x}(k) + u(k)^T R u(k) + 2\gamma_2(k)] \quad (5)$$

$$\text{subject to } \bar{x}(k+1) = A\hat{x}(k) + Bu(k) + \alpha_1(k), \quad \bar{x}(0) = \bar{x}_0, \text{ and } \bar{y}(k) = C\bar{x}(k) + \alpha_2(k),$$

where $\bar{x}(k), \hat{x}(k) \in \mathfrak{R}^n$ and $\bar{y}(k) \in \mathfrak{R}^p$ are the estimated random state sequences and measured output, respectively, $\alpha_1(k) \in \mathfrak{R}^n$, $\alpha_2(k) \in \mathfrak{R}^p$ and $\gamma_1, \gamma_2(k) \in \mathfrak{R}$ are discrete parameters, $A \in \mathfrak{R}^{n \times n}$ is state transition matrix, $B \in \mathfrak{R}^{n \times m}$ is control coefficient matrix and $G \in \mathfrak{R}^{n \times q}$ is noise coefficient matrix. The weighting matrices of performance index are represented by $S(N) \in \mathfrak{R}^{n \times n}$, $Q \in \mathfrak{R}^{n \times n}$ and $R \in \mathfrak{R}^{m \times m}$, respectively, with $S(N)$ and Q are positive semidefinite, and R is positive definite.

3. Principle of Model-Reality and Its Extension

Now, an expanded optimal control problem (EOP) which is equivalent to the SOP is defined as

$$\begin{aligned} \text{Minimize } J_E = & \frac{1}{2} \bar{x}(N)^T S(N) \bar{x}(N) + \gamma_1 + \sum_{k=0}^{N-1} [\frac{1}{2} \bar{x}(k)^T Q \bar{x}(k) + \frac{1}{2} u(k)^T R u(k) + \gamma_2(k) \\ & + \frac{1}{2} r_1 \|u(k) - v(k)\|^2 + \frac{1}{2} r_2 \|\bar{x}(k) - z(k)\|^2] \end{aligned} \quad (6)$$

subject to $\bar{x}(k+1) = A\hat{x}(k) + Bu(k) + \alpha_1(k)$, $\bar{x}(0) = \bar{x}_0$ and $\bar{y}(k) = C\bar{x}(k) + \alpha_2(k)$, together with

$$\left. \begin{aligned} \frac{1}{2}z(N)^T S(N)z(N) + \gamma_1 &= \varphi^*(z(N)) \\ \frac{1}{2}z(k)^T Qz(k) + v(k)^T Rv(k) + \gamma_2(k) &= L^*(z(k), v(k)) \\ Az(k) + Bv(k) + \alpha_1(k) &= f^*(z(k), v(k)) \\ Cz(k) + \alpha_2(k) &= h^*(z(k)) \end{aligned} \right\} \quad (7)$$

$$v(k) = u(k); \quad z(k) = \bar{x}(k) \quad (8)$$

Additional, the expectation of dynamic state equation is updated by the state estimation [4]:

$$\hat{x}(k) = \bar{x}(k) + K_f(k)(y(k) - \bar{y}(k)) \quad (9)$$

where

$$K_f(k) = M(k)C^T (CM(k)C^T + R_\eta)^{-1} \quad (10)$$

$$P(k) = M(k) - K_f(k)(CM(k)C^T + R_\eta)K_f(k)^T \quad (11)$$

$$M(k+1) = AP(k)A^T + GQ_\omega G^T, \quad M(0) = M_0. \quad (12)$$

Define the Hamiltonian function,

$$\begin{aligned} H(\cdot) &= \frac{1}{2}[u(k)^T Ru(k) + \bar{x}(k)^T Q\bar{x}(k)] + \gamma_2(k) \\ &+ p(k+1)^T [A\bar{x}(k) + Bu(k) + AK_f(k)(y(k) - C\bar{x}(k) - \alpha_2(k)) + \alpha_1(k)] \\ &+ q(k)^T [C\bar{x}(k) + \alpha_2(k) - \bar{y}(k)] - \lambda(k)^T u(k) - \beta(k)^T \bar{x}(k) \\ &+ \frac{1}{2} \|C\bar{x}(k) + \alpha_2(k) - \bar{y}(k)\|_{R_\eta}^2 + \frac{1}{2} r_1 \|u(k) - v(k)\|^2 + \frac{1}{2} r_2 \|\bar{x}(k) - z(k)\|^2 \end{aligned} \quad (13)$$

where $p(k), q(k) \in \mathfrak{R}^n$ (the costate), $\lambda(k) \in \mathfrak{R}^m$, and $\beta(k) \in \mathfrak{R}^n$ are Lagrange multipliers. Adjoining the constraints in (7) and (8), the augmented performance index is formed with applying (13). It is considered the increment of augmented performance index due to increments in all variables and assumed that the final time N is fixed. According to the Lagrange multiplier theory, this increment of augmented performance index should be zero at a constrained minimum [5]. Therefore, the following necessary optimality conditions are satisfied:

$$\begin{aligned} \text{(a)} \quad \nabla_{u(k)} H(\cdot) &= \mathbf{0}; \\ \Rightarrow u(k) &= -\bar{R}^{-1}(B^T p(k+1) - \bar{\lambda}(k)), \quad k \in [0, N-1] \end{aligned} \quad (14)$$

$$\text{where } \bar{R} = R + r_1 I_m; \quad \bar{\lambda}(k) = \lambda(k) + r_1 v(k).$$

$$\begin{aligned} \text{(b)} \quad \nabla_{x(k)} H(\cdot) - p(k) &= \mathbf{0}; \\ \Rightarrow p(k) &= \bar{Q}\bar{x}(k) + (A - AK_f(k)C)^T p(k+1) - \bar{\beta}(k), \quad p(N) = S(N)\bar{x}(N), \quad k \in [0, N-1] \end{aligned} \quad (15)$$

$$\text{where } \bar{Q} = Q + r_2 I_n; \quad \bar{\beta}(k) = \beta(k) + r_2 z(k).$$

$$\begin{aligned} \text{(c)} \quad \frac{\partial H(\cdot)}{\partial p(k+1)} - x(k+1) &= \mathbf{0}; \\ \Rightarrow \bar{x}(k+1) &= A\bar{x}(k) + Bu(k) + AK_f(k)(y(k) - C\bar{x}(k) - \alpha_2(k)) + \alpha_1(k), \quad \bar{x}(0) = \bar{x}_0, \quad k \in [0, N-1]. \end{aligned} \quad (16)$$

$$\text{(d)} \quad v(k) = u(k), \quad k \in [0, N-1]; \quad z(k) = \bar{x}(k), \quad k \in [0, N]. \quad (17)$$

The multipliers are calculated as

$$\lambda(k) = -\left(\frac{\partial f^*(\cdot)}{\partial v(k)} - B \right)^T \hat{p}(k+1) - \left(\frac{\partial L^*(\cdot)}{\partial v(k)} - Rv(k) \right) \quad (18)$$

$$\beta(k) = -\left(\frac{\partial f^*(\cdot)}{\partial z(k)} - (A - AK_f(k)C) \right)^T \hat{p}(k+1) - \left(\frac{\partial L^*(\cdot)}{\partial z(k)} - Qz(k) \right) \quad (19)$$

where $\hat{p}(k)$ is introduced as separation variable and the computation of parameters are

$$\left. \begin{aligned} \gamma_1 &= \varphi^*(z(N)) - \frac{1}{2}z(N)^T S(N)z(N) \\ \gamma_2(k) &= L^*(z(k), v(k)) - \frac{1}{2}[v(k)^T Rv(k) + z(k)^T Qz(k)] \\ \alpha_1(k) &= f^*(z(k), v(k)) - [Az(k) + Bv(k) + AK_f(k)[y(k) - Cz(k) - \alpha_2(k)]] \\ \alpha_2(k) &= h^*(z(k)) - Cz(k). \end{aligned} \right\} \quad (20)$$

The Hamiltonian function in (13) and the optimality of (14) – (16) are satisfied when the following modified model-based optimal control problem (MMOP) is solved,

$$\begin{aligned} \text{Minimize } J_{MM} = & \frac{1}{2} \bar{x}(N) S(N) \bar{x}(N) + \gamma_1 + \sum_{k=0}^{N-1} \left[\frac{1}{2} [u(k)^T R u(k) + \bar{x}(k)^T Q \bar{x}(k)] + \gamma_2(k) \right. \\ & - \lambda(k)^T u(k) - \beta(k)^T \bar{x}(k) \qquad \qquad \qquad \left. + \frac{1}{2} \|h(\bar{x}(k), \alpha_2(k)) - \bar{y}(k)\|_{R_\eta}^2 \right. \\ & \left. + \frac{1}{2} r_1 \|u(k) - v(k)\|^2 + \frac{1}{2} r_2 \|\bar{x}(k) - z(k)\|^2 \right] \end{aligned} \quad (21)$$

$$\text{subject to } \bar{x}(k+1) = A\bar{x}(k) + Bu(k) + AK_f(k)(y(k) - C\bar{x}(k) - \alpha_2(k)) + \alpha_1(k), \quad \bar{x}(0) = \bar{x}_0,$$

under specified $\alpha_1(k), \alpha_2(k), \gamma_1, \gamma_2(k)$, specified multipliers $\lambda(k), \beta(k)$, and specified $v(k), z(k)$.

4. Solution of Modified Model-Based Optimal Control Problem

The solutions of MMOP are the optimum of the SOP at the end of iteration procedures when the convergence is assumed to be achieved. These solutions are summarized in the following theorem.

Theorem 1

For a given modified model-based optimal control problem (MMOP) with satisfying the extended principle of model-reality differences, the following optimal solutions satisfy the necessary optimality conditions as in (14) – (16):

$$u(k) = -K(k)\bar{x}(k) + u_{ff}(k), \quad k \in [0, N-1] \quad (22)$$

$$p(k) = S(k)\bar{x}(k) + s(k), \quad p(k+1) = S(k+1)\bar{x}(k+1) + s(k+1), \quad k \in [0, N] \quad (23)$$

$$\bar{x}(k+1) = (A - AK_f(k)C - BK(k))\bar{x}(k) + Bu_{ff}(k) + AK_f(k)(y(k) - \alpha_2(k)) + \alpha_1(k), \quad k \in [0, N-1] \quad (24)$$

where

$$K(k) = [B^T S(k+1)B + \bar{R}]^{-1} B^T S(k+1) [A - AK_f(k)C] \quad (25)$$

$$S(k) = \bar{Q} + [A - AK_f(k)C]^T S(k+1) [A - AK_f(k)C - BK(k)], \quad S(N) = S_N, \quad k \in [0, N-1] \quad (26)$$

$$\begin{aligned} s(k) = & [A - AK_f(k)C - BK(k)]^T s(k+1) + K(k)^T \bar{\lambda}(k) - \bar{\beta}(k) \\ & + [A - AK_f(k)C - BK(k)]^T S(k+1) [AK_f(k)(y(k) - \alpha_2(k)) + \alpha_1(k)], \quad s(N) = \mathbf{0} \end{aligned} \quad (27)$$

$$u_{ff}(k) = [B^T S(k+1)B + \bar{R}]^{-1} [-B^T s(k+1) + \bar{\lambda}(k) - B^T S(k+1) [AK_f(k)(y(k) - \alpha_2(k)) + \alpha_1(k)]] \quad (28)$$

Proof

See [6].

5. Simulation Example

Consider an example of scalar system from [7]. We have defined the expectation of quadratic objective function and the SOP is given as follows:

$$\text{Minimize } J_S = E \left[\frac{1}{2} x(N)^T S(N) x(N) \right] + \frac{1}{2} \sum_{k=0}^{N-1} [x(k)^T Q x(k) + u(k)^T R u(k)]$$

$$\text{subject to } x(k+1) = Ax(k) + Bu(k) + G\omega(k), \quad x(0) = \bar{x}_0, \quad \text{and } y(k) = Cx(k) + \eta(k),$$

where $A = 0.9, B = 1, G = 1, N = 30, C = 1, x_0 = 1, S_N = 0, Q = 3$, and $R = 1$. The probability distributions of random state, process noise, and measurement noise are given as

$$x(0) \sim N(1, 0.1^2); \quad \omega(k) \sim N(0, 0.03^2); \quad \eta(k) \sim N(0, 0.1^2).$$

This problem has been discussed and its solution obtained by optimal smoothing was presented in [8].

Now, consider the following MMOP,

$$\text{Minimize } J_{MM} = \frac{1}{2} \bar{x}(N)^T S(N) \bar{x}(N) + \frac{1}{2} \sum_{k=0}^{N-1} [\bar{x}(k)^T Q \bar{x}(k) + u(k)^T R u(k)]$$

$$\text{subject to } \bar{x}(k+1) = A\bar{x}(k) + Bu(k) + AK_f(k)(y(k) - C\bar{x}(k) - \alpha_2(k)) + \alpha_1(k), \quad \bar{x}(0) = \bar{x}_0,$$

where $A = 1$, $B = 1$, $N = 30$, $S_N = 0$, $Q = 3$, $R = 1$. The parameters $r_1 = r_2 = 0$, and $k_v = k_z = k_p = 1$ are considered during the computations. We solve this problem by using the approach of model-reality differences in order to give the solution of real plant.

It is obvious that the state transition of model employed is different with real plant. The dynamic state constraint in the model is an estimation equation which obtained by Kalman filter. It gives the optimal state estimate, and under the separation principle, the optimal control law is generated in term of state estimate. The following graphical solutions are produced and the trajectories of state and control depict the steady-state solution. The optimal cost function is $J = 1.8442$ after convergence is achieving at third iterations and it is greater than the optimal cost function, $J = 1.8176$, of the deterministic equivalence of SOP [9].

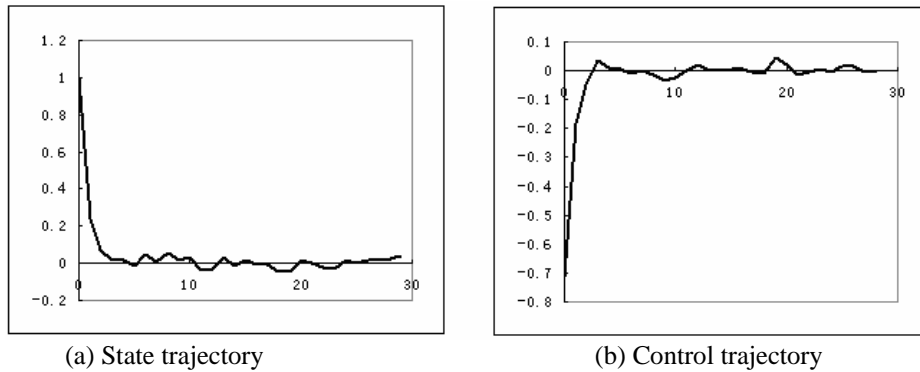


Fig. 1: Trajectory of state and control.

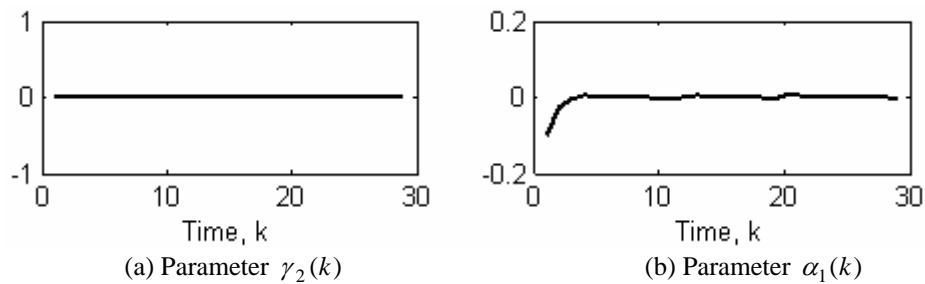


Fig. 2: Trajectories of parameters.

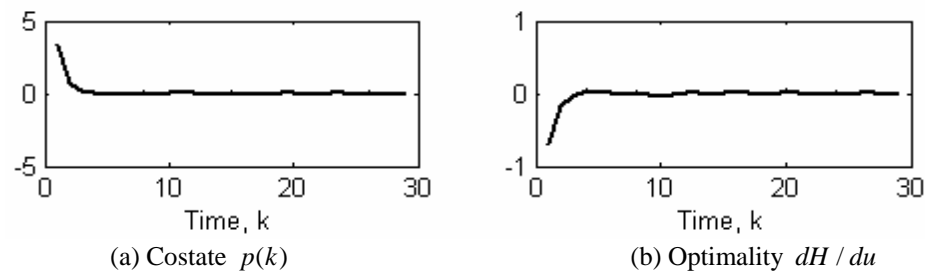


Fig. 3: Trajectories of costate and optimality.

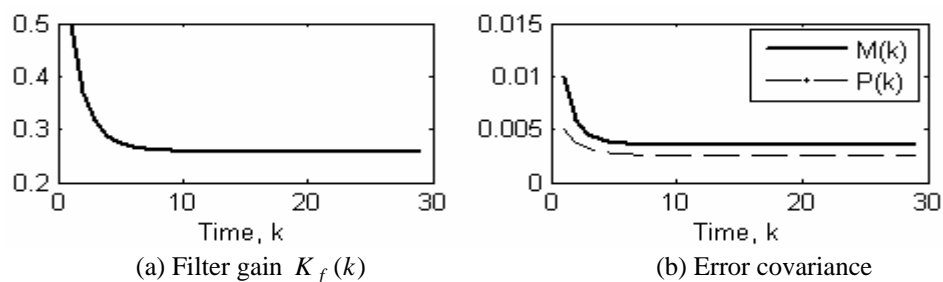


Fig. 4: Trajectories of filter gain and error covariance.

6. Conclusion

The extension of principle of model-reality differences has been made for solving the linear stochastic optimal control problem. Based on this principle, we have designed an integrated algorithm that takes into account the linear quadratic Gaussian as model-based problem. Since the real plant has different structure compare to the model used, the real optimum is most difficult to be obtained. With the proposed algorithm, we desire to give an alternative solution method in which the expenses can be reduced. The presented simulation example is simple and it is only a test run example to support the applicable of our algorithm. In conclusion, the proposed algorithm is effective and efficient to give the solution of linear stochastic optimal control problem. In the future, we shall focus on solving the nonlinear stochastic optimal control problems by using this proposed algorithm.

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8. References

- [1] Roberts, P.D. Optimal Control of Nonlinear Systems with Model-Reality Differences. *Proceeding of 31st IEEE Conference on Decision and Control*, 1992, Vol. 1, 257 – 258.
- [2] Becerra, V. M. *Development and Applications of Novel Optimal Control Algorithms*. City University, London: PhD Thesis, 1994.
- [3] Deisenroth, Marc. P., Toshiyuki Ohtsuka, Florian Weissel, Dietrich Brunn, and Uwe D. Hanebeck. Finite-Horizon Optimal State-Feedback Control of Nonlinear Stochastic Systems Based on a Minimum Principle. *6th IEEE International Conference on Multisensor Fusion and Integration for Intelligent Systems (MFI 2006)*, pages 371–376, Heidelberg, Germany, September 2006.
- [4] Bryson, A. E., and Ho, Y. C. *Applied Optimal Control*. Hemisphere Publishing Corporation, 1975.
- [5] Lewis, F., L. *Optimal Control*. New York: Wiley, 1986.
- [6] Kek, S.L. and Mohd Ismail, A.A. (2009b). *Discrete Linear Stochastic Optimal Control based on Extended Model-Reality Differences Principle*. Universiti Teknologi Malaysia: Technical Report N0. 6, 2009b.
- [7] Bryson, A. E. *Applied Linear Optimal Control*. Cambridge University Press, 2002.
- [8] Kek, S. L. and Ismail, M. A. A. Application of Optimal Smoothing in Discrete Linear Stochastic Control Problem with State Feedback. *Proceeding of 3rd International Conference on Mathematical Sciences (ICoMS2008)*, Institut Pertanian Bogor, Indonesia. (05-06 August, 2008, Institut Pertanian Bogor, will be appeared).
- [9] Kek, S.L. and Mohd Ismail, A.A. *Discrete Linear Quadratic Gaussian Problem Using Extended Deterministic DISOPE Approach*. Universiti Teknologi Malaysia: Technical Report No. 5, 2009a