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The Number of the Two Dimensional Run Length Constrained Arrays

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Abstract. First, a new framework describing the transfer matrices for the two dimensional run length constrained arrays (codes) is introduced, and some important properties of the transfer matrices T_m $(m \geq 1)$ are derived in this framework. Then, using these properties, it is shown that the numbers $N(m,n)$ $(m,n \geq 1)$ of the two dimensional binary arrays satisfying the $(1,\infty)$ - run length constraint is expressible by a linear recurrence equation of a fixed order, approximately equal to $\dim(T_m)/2$. Finally, to demonstrate the effectiveness of the result obtained, some numerical examples are presented.

Keywords: Run length constrained array, transfer matrix, information capacity; data storage

1. Introduction

Run length constrained arrays (codes) are widely used in digital data recording and transmission. Its generalization to the two-dimensional case is of potential interest in the page-oriented information storage technologies, such as holographic storage. A one-dimensional binary sequence is said to satisfy a (d,k)*-run length constraint* if every run of zeros in the sequence has length at least d and at most k . Similarly an 2 -dimensional binary array is said to satisfy a (d,k)-run length constraint if it satisfies the one-dimensional -run length constraint both horizontally and vertically. Let the number of 2 -dimensional binary arrays of size $m \times n$ satisfying the (d,k)-run length constraint be denoted by $N_{d,k}(m,n)$. Then the *information capacity* is defined as

$$C_{d,k} = \lim_{m,n\to\infty} \frac{1}{mn} \log_2 N_{d,k}(m,n) \tag{1}$$

which may be interpreted as the maximum number of bits of information that can be stored asymptotically per unit volume. Recently a great deal of effort has been made for evaluating the numbers $N_{d,k}(m,n)$ $(m,n \geq 1)$ and the limit $C_{d,k}$, in particular, for investigating the existence of the limit and lower and upper bounds on $C_{d,k}$ for various values of (d,k) . See, e.g., [1]-[6] and the references herein.

This paper focuses on a special case of the 2-dimensional constrained arrays with $d = 1$ and $k = \infty$. For this case, the number $d_m := N_{1,\infty}(m,1)$ is well known as the *Fibonacci number*. Further it is easily seen by exchanging the roles of 0 and 1 that $C_{1,\infty} = C_{0,1}$. For notational simplicity, hereafter $N_{1,\infty}(m,n) = N_{0,1}(m,n)$ will be denoted simply by $N(m,n)$.

This special case was extensively studied in [1]-[4] and also in [5], and many important results have been obtained. One of important and interesting results among them is the work by Calkin and Wilf [1]. In this work, they used the fact that the number $N(m,n)$ can be expressed in the form

$$N(m,n) = \mathbf{1}^t T_m^{n-1} \mathbf{1}, \quad m,n \geq 1 \tag{2}$$

where $\mathbf{1}$ indicates the column vector with all entries equal to 1 and a compatible dimension, $\mathbf{1}^t$ denotes the transpose, T_m is the *transfer matrix* of the two-dimensional $(1,\infty)$ - run length constraint problem introduced in [1]. The transfer matrix T_m is a symmetric $d_m \times d_m$ matrix with all entries equal to 0 or 1 and the order d_m is computed in the following recursive manner:

$$\begin{cases} d_{-1} = d_0 = 1 \\ d_m = d_{m-1} + d_{m-2}, \quad m \geq 1. \end{cases} \tag{3}$$

Therefore all the properties of $N(m,n)$ can be obtained through the corresponding transfer matrix T_m.

$N(m,n)$ even for relatively large values of n and m. Finally to demonstrate effectiveness of our results, the recursive equations of $N(m,n)$ for $m = 2,3,4$ will be computed as numerical examples.

2. Description of Transfer Matrices

In this section, we first introduce a new framework for representing the transfer matrix T_m defined in Calkin and Wilf [1], and investigate their important properties. First let $\{0,1\}^{p\times q}$ denote the set of all $p\times q$ -dimensional matrices with all entries in $\{0,1\}$. Then, we have the following results.

LEMMA 1. Let us introduce the two sequences of matrices $V^{(m)}, W^{(m)} \in \{0,1\}^{d_m \times m}$ $(m \geq 1)$ by the following recursive formula:

$$V^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad V^{(m)} = \begin{bmatrix} V^{(m-1)} & \mathbf{0} \\ (V^{(m-1)})^{[d_{m-2}]} & \mathbf{1} \end{bmatrix}, m \geq 2, \qquad W^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad W^{(m)} = \begin{bmatrix} \mathbf{0} & W^{(m-1)} \\ \mathbf{1} & (W^{(m-1)})^{[d_{m-2}]} \end{bmatrix}, m \geq 2 \tag{4}$$

where the notation $X^{[k]}$ indicates the sub-matrix composed of the first k rows of a matrix X, and $\mathbf{0}$ is a zero column vector with a compatible dimension. Further let us represent the matrices $V^{(m)}, W^{(m)}$ as the row vectors:

$$V^{(m)} = \begin{bmatrix} v_1^{(m)} \\ \vdots \\ v_{d_m}^{(m)} \end{bmatrix} \in \{0,1\}^{d_m \times m}, \quad W^{(m)} = \begin{bmatrix} w_1^{(m)} \\ \vdots \\ w_{d_m}^{(m)} \end{bmatrix} \in \{0,1\}^{d_m \times m}$$

Then the following statements hold.

(i) The row vectors $v_1^{(m)}, \cdots, v_{d_m}^{(m)}$ are all distinct and so are $w_1^{(m)}, \cdots, w_{d_m}^{(m)}$.

(ii) For each $i = 1, \cdots, d_m$, $v_i^{(m)}$ and $w_i^{(m)}$ coincide in the reversed order, that is, if

$v_i^{(m)} = [v_{i,1}^{(m)} \quad v_{i,2}^{(m)} \quad \cdots \quad v_{i,m}^{(m)}]$ then $w_i^{(m)} = [v_{i,m}^{(m)} \quad v_{i,m-1}^{(m)} \quad \cdots \quad v_{i,1}^{(m)}]$.

(iii) For each $i = 1, \cdots, d_m$, both $v_i^{(m)}$ and $w_i^{(m)}$ have no two consecutive 1's in their components, respectively.

(iv) Set $\{v_1^{(m)}, \cdots, v_{d_m}^{(m)}\}$ coincides with the set of all possible row vectors in $v \in \{0,1\}^{1\times m}$ having no two consecutive 1's.

Next, let $p, q \geq 1$ be arbitrary integers, and for any two matrices

$$V = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} \in \{0,1\}^{p\times q}, \quad W = \begin{bmatrix} w_1 \\ \vdots \\ w_p \end{bmatrix} \in \{0,1\}^{p\times q},$$

define the two operations $\wedge$ and $\vee$ by

$$\begin{cases} V \vee W := \left[v_i \vee w_j \right]_{i,j=1}^{p} \in \{0,1\}^{p\times p} \\ V \wedge W := \left[v_i \wedge w_j \right]_{i,j=1}^{p} \in \{0,1\}^{p\times p} \end{cases} \quad \text{where } v_i \vee w_j := \begin{cases} 1 \text{ if } \left\langle v_i, w_j \right\rangle = 0 \\ 0 \text{ otherwise.} \end{cases}, \quad v_i \wedge w_j := \begin{cases} 1 \text{ if } v_i = w_j, \\ 0 \text{ otherwise.} \end{cases}$$

Then the following theorem can be proved. The statement (ii) has been shown in a different framework in [6], but it would be worthwhile to reprove it because our framework gives a simpler and more straightforward proof.

THEOREM 1. The following facts hold.

(i) The transfer matrix T_m is expressed as $T_m = V^{(m)} \vee V^{(m)} = W^{(m)} \vee W^{(m)}$.

(ii) Further T_m is computed by the following recursive formula:

$$T_m = \begin{bmatrix} T_{m-1} & \left(T_{m-1}^{[d_{m-2}]}\right)^t \\ T_{m-1}^{[d_{m-2}]} & \mathbf{0} \end{bmatrix}, \quad T_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{5}$$

Proof. The statement (i) can be verified simply by observing that Lemma 1 ensures that the vectors $v_1^{(m)},\cdots,v_{d_m}^{(m)}$ and $w_1^{(m)},\cdots,w_{d_m}^{(m)}$ satisfy the conditions for the transfer matrix T_{m-1} defined in [1].

To prove (ii), letting $k \geq 2$ and assuming that (11) is true for $m = k-1$, it will be proved that (11) is also true for $m = k$. This is done by directly evaluating $V^{(k)} \vee V^{(k)}$. □

LEMMA 2. Consider two matrices V, W and a permutation matrix P represented as

$$\begin{cases} V = \begin{bmatrix} v_1 \\ \vdots \\ v_q \end{bmatrix} \in \{0,1\}^{q\times r}, \quad W = \begin{bmatrix} w_1 \\ \vdots \\ w_q \end{bmatrix} \in \{0,1\}^{q\times r} \\ P = [p_1 \quad \cdots \quad p_q] \in \{0,1\}^{q\times q} \end{cases} \tag{6}$$

where v_i, w_i are row vectors and each p_i is a column unit vector. Then the following equality holds true:

$$(PV) \vee (PW) = P(V \vee W)P. \tag{7}$$

Proof: First note that P is represented as a permutation of the identity matrix $I = [e_1 \quad \cdots \quad e_q]$ where e_k is the column unit vector with k-th entry equal to 1. Therefore it suffices to verify (6) for a special case such that P is a permutation matrix of the form

$$P = [e_1 \quad \cdots \quad e_q]_{\text{c}:i\leftrightarrow j}, \quad 1 \leq i \leq j \leq q$$

where $[\cdots]_{\text{c::}i\leftrightarrow j}$ indicates an interchange of the *i*-th and *j*-th column vectors. For this permutation matrix P, we have

$$(PV) \vee (PW) = V_{\text{r}:i\leftrightarrow j} \vee W_{\text{r}:i\leftrightarrow j} = \begin{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_q \end{bmatrix}_{\text{r}:i\leftrightarrow j} \vee \begin{bmatrix} w_1 \\ \vdots \\ w_q \end{bmatrix}_{\text{r}:i\leftrightarrow j} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_q \end{bmatrix} \vee \begin{bmatrix} w_1 \\ \vdots \\ w_q \end{bmatrix} \end{bmatrix}_{\substack{\text{c}:i\leftrightarrow j \\ \text{r}:i\leftrightarrow j}} = P(V \vee W)P$$

where similarly $[\cdots]_{\text{r}:i\leftrightarrow j}$ indicates the interchange of the *i*-th and *j*-th column vectors. □

THEOREM 2. Let $V^{(m)}, W^{(m)}$ be given by (4) and (5), and define

$$P_m := V^{(m)} \wedge W^{(m)} \in \{0,1\}^{d_m \times d_m}, \quad m \geq 1. \tag{8}$$

Then, for any $m \geq 1$ the following statements hold.

P_m is an $d_m \times d_m$ symmetric permutation matrix, and satisfies $V^{(m)} = P_m W^{(m)}$. Further $T_m^n P_m = P_m T_m^n, \quad n = 0, \pm 1, \pm 2, \cdots$.

3. Recursive Formulas

In this section, it is shown that the numbers $N(m,n)\,(m,n \geq 1)$ of the two-dimensional binary arrays satisfying the $(1,\infty)$ - run length constraint can be represented by a linear recurrence equation of a fixed order. In particular, it is shown that the order of the linear recurrence equation is approximately equal to $\dim(T_m)/2$ for large values of m.

LEMMA 3. Let P_m be given by (14). Then, its trace is given as

$$\text{tr}(P_m) = \begin{cases} d_{(m-2)/2} & \text{if} \quad m \text{ is even} \\ d_{(m+1)/2} & \text{if} \quad m \text{ is odd}. \end{cases} \tag{9}$$

Proof. It follows from (14) that $\text{tr}(P_m) = \sum_{i=1}^{d_m} (v_i^{(m)} \wedge w_i^{(m)})$

First note that, by virtue of the definition of operation $\wedge$, $v_i^{(m)} \wedge w_i^{(m)} = 1$ if and only if $v_i^{(m)} = w_i^{(m)}$. In addition to this, since by (ii) of Lemma 1 $v_i^{(m)}$ and $w_i^{(m)}$ coincide in the reversed order, therefore if $v_i^{(m)} \wedge w_i^{(m)} = 1$ then $v_i^{(m)}$ must be of the form

$$v_i^{(m)} = \left[v_{i,1}^{(m)} \cdots v_{i,m/2-1}^{(m)} \; v_{i,m/2}^{(m)} \; v_{i,m/2}^{(m)} \; v_{i,m/2-1}^{(m)} \cdots v_{i,1}^{(m)} \right] \tag{10}$$

when m is even, and

$$v_i^{(m)} = \left[v_{i,1}^{(m)} \cdots v_{i,(m-1)/2}^{(m)} \; v_{i,(m-1)/2+1}^{(m)} \; v_{i,(m-1)/2}^{(m)} \cdots v_{i,1}^{(m)} \right] \tag{11}$$

when m is odd.

Furthermore, since $v_i^{(m)}$ has no two consecutive 1's by Lemma 1, it is easily seen from (20) and (21) that a necessary and sufficient condition for $v_i^{(m)} \wedge w_i^{(m)} = 1$ is that $v_i^{(m)}$ is of the form

$$v_i^{(m)} = \left[v_{i,1}^{(m)} \cdots v_{i,(m-2)/2}^{(m)} \;\; 0 \;\; 0 \;\; v_{i,(m-2)/2}^{(m)} \cdots v_{i,1}^{(m)} \right] \qquad m \text{ is even} \tag{12}$$

$$v_i^{(m)} = \left[v_{i,1}^{(m)} \cdots v_{i,(m-1)/2}^{(m)} \; v_{i,(m+1)/2}^{(m)} \; v_{i,(m-1)/2}^{(m)} \cdots v_{i,1}^{(m)} \right] \qquad m \text{ is odd} \tag{13}$$

Now (9) is evaluated. First, for the case of m being even, the summation is equal to the number of all possible vectors that has the form of (11), which in turn, the form of $\left[v_{i,1}^{(m)} \cdots v_{i,(m-2)/2}^{(m)} \right]$, and hence the summation is equal to $d_{(m-2)/2}$. Similarly, for the case of m being odd, the summation (19) is equal to the number of all possible vectors that has the form of (12), which in turn, the form of $\left[v_{i,1}^{(m)} \cdots v_{i,(m+1)/2}^{(m)} \right]$, and hence the summation is equal to $d_{(m-1)/2}$. This completes the proof. □

Next, let $m \ge 1$ be arbitrarily fixed, and define a sequence of d_m-dimensional column vectors by

$$\varphi_k := \begin{bmatrix} \varphi_{k1} \\ \varphi_{k2} \\ \vdots \\ \varphi_{kd_m} \end{bmatrix} := T_m^{k-1} \mathbf{1}, \;\; k \ge 1,$$

and a sequence of matrices

$$\Phi_k = \begin{bmatrix} \varphi_k(1) \\ \vdots \\ \varphi_k(d_m) \end{bmatrix} := [\varphi_1 \;\; \varphi_2 \;\; \cdots \;\; \varphi_k] = [\mathbf{1} \quad T_m \mathbf{1} \quad \cdots \quad T_m^{k-1}\mathbf{1}], \;\; k \ge 1. \tag{14}$$

LEMMA 4. The following statements hold:

(i) The vectors φ_k defined in (13) satisfy $P_m \varphi_k = \varphi_k$, $\quad k \ge 1$

(ii) The matrices Φ_k defined in (14) satisfy $\operatorname{rank}\Phi_k = \operatorname{rank}\Phi_{k+1} \Rightarrow \operatorname{rank}\Phi_k = \operatorname{rank}\Phi_{k+\alpha}, \quad \forall \alpha \ge 2$.

Proof. The statement (i) can be easily proved using Theorem 2 (ii). In fact, for any $k \ge 1$

$$P_m \varphi_k = P_m T_m^k \mathbf{1} = T_m^k P_m \mathbf{1} = T_m^k \mathbf{1} = \varphi_k .$$

Next to prove (ii), assume $\operatorname{rank}\Phi_k = \operatorname{rank}\Phi_{k+1}$. Then, first note that $\operatorname{rank}\Phi_k = \operatorname{rank}\Phi_{k+1}$ implies that φ_k can be expressed as a linear combination of the vectors in Φ_k, i.e., $\varphi_k = \Phi_k \xi$ for some vector $\xi \in \boldsymbol{R}^k$. This fact leads to

$$\varphi_{k+1} = T_m \varphi_k = T_m \Phi_k \xi = T_m [\varphi_0 \;\; \varphi_1 \;\; \cdots \;\; \varphi_{k-1}] \xi = [\varphi_1 \;\; \cdots \;\; \varphi_{k-1} \;\; \Phi_k \xi] \xi = \Phi_k [e_2 \;\; \cdots \;\; e_k \;\; \xi] \xi.$$

Therefore φ_{k+1} is also a linear combination of $\varphi_0, \varphi_1, \cdots, \varphi_{k-1}$, and hence $\operatorname{rank}\Phi_k = \operatorname{rank}\Phi_{k+1} = \operatorname{rank}\Phi_{k+2}$. Repeating this process, one can verify $\operatorname{rank}\Phi_k = \operatorname{rank}\Phi_{k+\alpha}$ for all $\alpha \ge 2$.

LEMMA 5. Let Φ_k be the matrices defined by (14). Then, $\max_{k \ge 0} \operatorname{rank}\Phi_k \le \frac{1}{2}\{d_m + \operatorname{tr}(P_m)\} =: r_m$

Proof. First, recall that P_m is a symmetric permutation matrix and $\operatorname{tr}(P_m)$ is given by (18). It is not difficult to see from LEMMA 4 (i) that, for every $k \ge 1$, $\operatorname{tr}(P_m)$ entries in the vector φ_k are fixed by this permutation P_m and the other entries in φ_k are pair-wisely permuted but the two entries in each pair are equal. That is to say, there exist two subsets $\{\alpha_1, \cdots, \alpha_q\}$ and $\{\beta_1, \cdots, \beta_p, \gamma_1, \cdots, \gamma_p\}$ of $\{1, \cdots, d_m\}$ with $q := \operatorname{tr}(P_m)$ and $p := (d_m - q)/2$, independent of $k \ge 1$, such that the entries $\varphi_{k\alpha_i}, \cdots, \varphi_{k\alpha_q}$ of φ_k are not permuted by P_m and the other entries are pair-wisely equal, i.e., $\varphi_{k\beta_1} = \varphi_{k\gamma_1}, \cdots, \varphi_{k\beta_p} = \varphi_{k\gamma_p}$. Therefore, the row vectors of Φ_k given in (14) satisfy

$$\varphi_k(\beta_1) = \varphi_k(\gamma_1), \cdots, \varphi_k(\beta_p) = \varphi_k(\gamma_p), \quad \forall k \geq 0 .$$

Therefore there are at least p linearly dependent vectors in $\{\varphi_k(1), \cdots, \varphi_k(d_m)\}$, and hence

$\max_{k\geq 0} \mathrm{rank}\Phi_k \leq d_m - \frac{1}{2}\{\mathrm{d}_m - \mathrm{tr}(P_m)\} = \frac{1}{2}\{\mathrm{d}_m + \mathrm{tr}(P_m)\}$, which proves the desired result.

Now it is ready to state and prove our main theorem as follows.

THEOREM 3. Let $m \geq 1$ be arbitrarily fixed. Consider the numbers $N(m,n)$ given by (2) and the vectors φ_n given by (14), that is,

$$\begin{cases} N(m,n) = \mathbf{1}^t T_m^{n-1}\mathbf{1}, & m,n \geq 1 \\ \varphi_n := T_m^{n-1}\mathbf{1}, & n \geq 1. \end{cases} \tag{15}$$

Then the following statements hold:

(i) The sequence $\{\varphi_n\}$ satisfies a linear recurrence equation of order r_m, that is, $\varphi_n = \beta_1\varphi_{n-1} + \beta_2\varphi_{n-2} + \cdots + \beta_{r_m}\varphi_{n-r_m}$, $n > r_m$ where β_k $(k = 1, \cdots, r_m)$ are some constants, independent of n.

(ii) The numbers $N(m,n) = \mathbf{1}^t T_m^{n-1}\mathbf{1}$ $(n \geq 1)$ also satisfy the same linear recurrence equation as (28), that is,

$$N(m,n) = \beta_1 N(m,n-1) + \beta_2 N(m,n-2) + \cdots + \beta_{r_m} N(m,n-r_m), \quad n > r_m \tag{16}$$

Proof. This theorem is easily proved. In fact, by virtue of LEMMA 4, φ_n with $n \geq r_m$ is expressed as a linear recursive equation of the form (28), which can be also written as

$$T_m^{n-1}\mathbf{1} = \beta_1 T_{d_m}^{n-2}\mathbf{1} + \beta_2 T_m^{n-3}\mathbf{1} + \cdots + \beta_{r_m} T_m^{n-1-r_m}\mathbf{1}, \quad n > r_m \tag{17}$$

and since $N(m,k) = \mathbf{1}^t T_m^{k-1}\mathbf{1}$ ($k \geq 1$), the linear recurrence equation (16) directly follows form (30).

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