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Theoretical Analysis of Inverse Generalized Exponential Models

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Abstract. This article presents the theoretical analysis of Inverse Generalized Exponential Models. The shapes of the properties of the Inverse Generalized Exponential distribution are discussed. Inverse moments of Inverse Generalized Exponential distribution are derived. In this article we have presented the relationship between shape parameter and other properties such probability distribution, distribution function, reliability function, hazard function and cumulative hazard function, mean, median, mode, variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis models are presented graphically and mathematically. Here we compare these relevant parameters such as shape, scale parameters by using Monte carol simulation.

Keywords: Inverse Generalized Exponential distribution, Simulation analysis, Properties.

1. Introduction

The Inverse Generalized Exponential models are the reliability models can be used in the reliability engineering discipline. We have developed this new reliability model and are the generalization of the inverse exponential distribution. The Inverse Generalized Exponential distribution approaches to the inverse exponential distribution when $\beta = 1$ and $t_0 = 0$. The Inverse Generalized Exponential models can be used as a standard in reliability for modeling time-dependent failure data. The Inverse Generalized Exponential distribution can be used to model a variety of failure characteristics such as infant mortality, random failures, wear-out, and failure-free periods. The Inverse Generalized Exponential distribution can also be used to determine the cost effectiveness and maintenance periods of reliability-centered maintenance activities. This paper focuses on the Theoretical analysis of the Inverse Generalized Exponential distribution's to model in which some operational time has already been accumulated for the equipment of interest. This paper presents the relationship between shape parameter and other properties such as $f_{\text{IGE}}(t), F_{\text{IGE}}(t), R_{\text{IGE}}(t), h_{IGE}(t), H_{\text{IGE}}(t)$. From the inverse moment estimation of the Inverse Generalized Exponential distribution's we derive the mean, median, mode, variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis models are presented graphically and mathematically. The Inverse Generalized Exponential distribution will be suitable for modeling for the applications of agriculture, botany, economics, medicine, psychology, zoology, life testing and reliability mechanical or electrical components lying in the life testing experiment. Some works has been done on Generalized Exponential distribution by Gupta and Kundu (2003). On the same pattern some works has already been done on weibull distribution by Liu (1997), Gupta and Kundu (1999, 2001, 2003).

This paper considers a three parameter Inverse Generalized Exponential distribution. In section 2, we have discussed some of its distributional properties graphically presented. In section 3, we have presented the inverse moment estimation of Inverse Generalized Exponential distribution and discussed its distributional properties graphically and mathematically presented.

2. Inverse Generalized Exponential Models Analysis

2.1. Inverse Generalized Exponential Probability Distribution

The Inverse Generalized Exponential probability distribution has three parameters β, η and t_0. It can be used to represent the failure probability density function (PDF) and is given by:

$$f_{IGE}(t) = \frac{\beta}{\eta}\left(\frac{1}{t-t_0}\right)^2 Exp\left[-\frac{1}{\eta}\left(\frac{1}{t-t_0}\right)\right]^{\beta} \quad , \quad \eta > 0, \beta > 0, t_0 > 0, -\infty < t_0 < t \tag{2.1}$$

Where β is the shape parameter representing the different pattern of the Inverse Generalized Exponential PDF and is positive and η is a scale parameter representing the characteristic life at which $(36.8)^{\beta}\%$ of the population can be expected to have failed and is also positive, t_0 is a location or shift or threshold parameter (sometimes called a guarantee time, failure-free time or minimum life). If $t_0 = 0$ then the Inverse Generalized Exponential distribution is said to be two-parameter Inverse Generalized Exponential distribution. In the life testing process if $t_0 > 0$ then the origin of the PDF lies to the left of the PDF of the recorded life time data. For the recorded life time data when $t_0 < 0$ then the origin of the PDF of Inverse Generalized Exponential distribution lies to the right of the PDF. In practical terms which may be explained as there being some delay before the duty actually starts. It is important to note that the restrictions in eq. (2.1) on the values of t_0, η, β are always the same for the Inverse Generalized Exponential distribution.

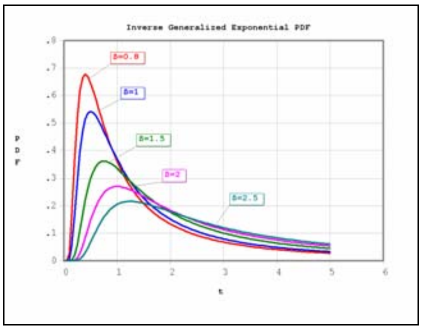

Fig 2.1 The Inv Gen Expo PDF

Fig. 2.1 shows the diverse shape of the Inverse Generalized Exponential PDF with $t_0 = 0$ and value of $\eta = 1$ and β (=0.8, 1, 1.5, 2, 2.5). Fig. 2.1 shows the diverse shape of the Inverse Generalized Exponential PDF which is quite similar with the Inverse weibull probability density functions. The Inverse Generalized Exponential and the Inverse weibull distributions are both the generalization of an inverse exponential distribution.

2.2. Cumulative Distribution Function

The cumulative distribution function (CDF) of Inverse Generalized Exponential distribution is denoted by $F_{IGE}(t)$ and is defined as

$$F_{IGE}(t) = Exp\left[-\frac{1}{\eta}\left(\frac{1}{t-t_0}\right)\right]^{\beta} \tag{2.2}$$

When the CDF of the Inverse Generalized Exponential distribution has zero value then it represents no failure components by t_0. When $t = t_0 + \eta$ then $F_{IGE}(t_0 + \eta) = \left(e^{-\frac{1}{\eta}}\right)^{\beta}$ and for $\eta = 1$ then it also gives $F_{IGE}(t_0 + 1) = (e^{-1})^{\beta} = (0.3678)^{\beta}$, it represents the characteristic life' or 'characteristic value. Fig. 2.2 shows the special case of Inverse Generalized Exponential CDF with $t_0 = 0$ and for the value of $\eta = 1$ and β (=0.8, 1, 1.5, 2, 2.5). It is clear from the Fig. 2.2 that all curves started from the point of origin the characteristic point for the Inverse Generalized Exponential CDF.

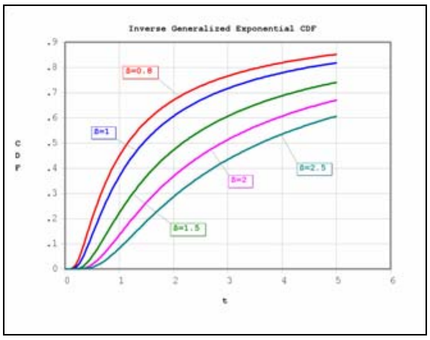

Fig. 2.2 The Inv Gen Expo CDF

2.3. Reliability Function

The reliability function (RF), denoted by $R_{IGE}(t)$ (also known as the survivor function) is defined as 1-$F_{IGE}(t)$

$$R_{IGE}(t) = 1 - Exp\left[-\frac{1}{\eta}\left(\frac{1}{t - t_0}\right)\right]^{\beta} \tag{2.3}$$

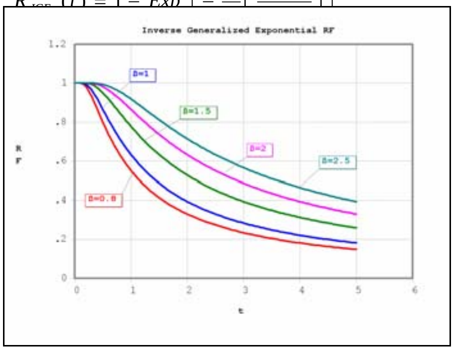

Fig. 2.3 The Inv Gen Expo RF

We see that $R_{IGE}(t)$ + $F_{IGE}(t)$ = 1. Fig. 2.3 shows the Inverse Generalized Exponential RF with t_0 = 0 and value of $\eta = 1$ and β (=0.8, 1, 1.5, 2, 2.5). It is clear that all curves intersect at the point of one the characteristic point for the Inverse Generalized Exponential $R_{IGE}(t)$.

2.4. Hazard Function

The hazard function (HF) (also known as instantaneous failure rate) denoted by $h_{IGE}(t)$ and is defined as $f_{IGE}(t)/R_{IGE}(t)$

$$h_{IGE}(t) = \frac{\frac{\beta}{\eta}\left(\frac{1}{t - t_0}\right)^2 Exp\left[-\frac{1}{\eta}\left(\frac{1}{t - t_0}\right)\right]^{\beta}}{1 - Exp\left[-\frac{1}{\eta}\left(\frac{1}{t - t_0}\right)\right]^{\beta}} \tag{2.4}$$

When β = 1, the distribution is the same as the inverse exponential distribution for a constant hazard function so the inverse exponential distribution is a special case of the Inverse Generalized Exponential distribution and the Inverse Generalized Exponential distribution can be treated as a generalization of the inverse exponential distribution. When β < 1, the hazard function is continually decreasing which represents early failures. When β > 1, the hazard function is continually increasing which represents wear-out failures. So the Inverse Generalized Exponential is a very flexible distribution. It can be used to represent a wide variety of in-service life failure patterns for many types of products. Fig. 2.4 shows the Inverse Generalized Exponential HF with t_0 = 0 and value of $\eta = 1$ and β (=0.8, 1, 1.5, 2, 2.5).

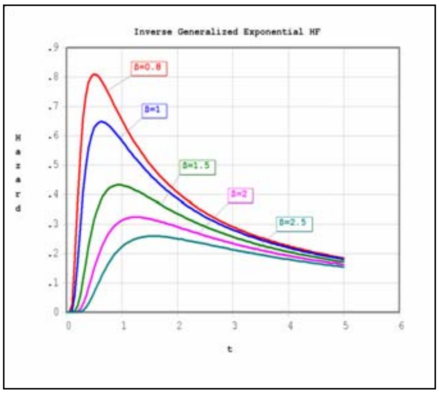

Fig. 2.4 The Inv Gen Expo HF

2.5. Cumulative Hazard Function

The cumulative hazard function (CHF), denoted by $H_{IGE}(t)$ and is defined as

$$H_{IGE}(t) = -\ln\left|1 - Exp\left[-\frac{1}{\eta}\left(\frac{1}{t-t_0}\right)\right]^{\beta}\right| \tag{2.5}$$

The relationships between CDF and CHF of the Inverse Generalized Exponential distribution are represented as

$$F_{IGE}(t) = 1 - e^{-H_{IGE}(t)} \quad \text{or} \quad H_{IGE}(t) = -\ln[1 - F_{IGE}(t)]$$

The Inverse Generalized Exponential CHF with $t_0 = 0$ and $\eta = 1$ and β (=0.8, 1, 1.5, 2, 2.5). It is important to note that the entire curve started through the points of origin on the graph of Inverse Generalized Exponential CHF.

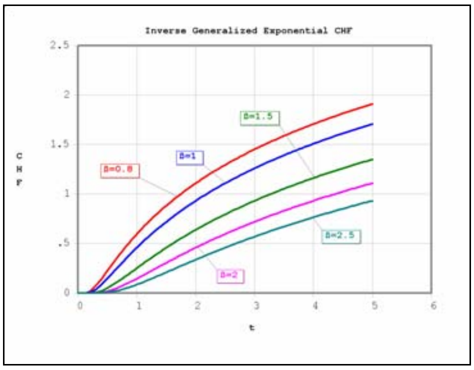

Fig. 2.5 The Inv Gen Expo CHF

2.6. Inverse Moments Estimation of Inverse Generalized Exponential model

The Inverse Moments Estimation of Inverse Generalized Exponential model is defined as

$$\mu'_{r-1} = \frac{\eta^r}{\beta^r}\Gamma(1+r), \quad r = 1,2,3,4,...... \tag{2.6}$$

The inverse Moments Estimator is helpful for finding the properties of the Inverse Generalized Exponential distribution. The method of inverse Moments Estimator for the parameters is

$$m'_{r-1} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{t^r{}_i}$$

3. Inverse Generalized Exponential Models and Simulation Analysis

Here every Inverse Generalized Exponential model is in a form of simulation analysis. The process of designing the Inverse Generalized Exponential models and then we have conducting computer-based experiments with these models to describe, explain and predicting the different patterns of these models over extended periods of real time. Other important properties of the Inverse Generalized Exponential distribution are summarized as follows. It is important to note that figures 3.1 to 3.4 are all based on the assumptions that minimum life is zero.

3.1. Mean Life

The mean life of the Inverse Generalized distribution also known as mean-time-to-failure (MTTF) and is defined as

$$Mean_{IGE} = \frac{\eta}{\beta} \tag{3.1}$$

From our calculation it is clear that there is no mean life when $\beta < 0$. It is important to note that the maximum value of the Mean/η is 10 when $\beta \cong$ 0.1. Here the Gamma functions $\Gamma(1+r)$ in the inverse moment function which can be calculated with Lanczos' approximate formula Lanczos (1964). The relationship between β and the mean life/η is shown in Fig. 3.1.Tables of the gamma function can be found in Bohoris (1994) and Kececioglu (1991). For the convenience of display we substitute this function in this notation γ_k.

$$\gamma_r = \Gamma(1 + r) \quad r = 1,2,3,4,\ldots\ldots \tag{3.1a}$$

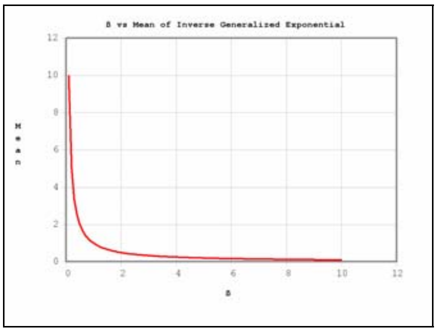

Fig 3.1 β vs mean life/η

3.2. Median Life

The median life (50^{th} percentile) of the Inverse Generalized Exponential distribution is defined as

$$Median_{IGE} = t_0 + \frac{1}{\eta}\left[\ln(2)^{\frac{1}{\beta}}\right]^{-1} \tag{3.2}$$

This is the life by which 50% of the units will be expected to have failed, and so it is also the life at which 50% of the units would be expected to still survive The relationship between β and (median life/η) is shown in Fig. 3.2. Taking the first derivative of equation (3.2) and equating it to 0, an extremely large value can be obtained: as $\beta \to \infty$, median life/$\eta \to$ 1. Fig. 3.1shows that β and median life/η have a positive proportion.

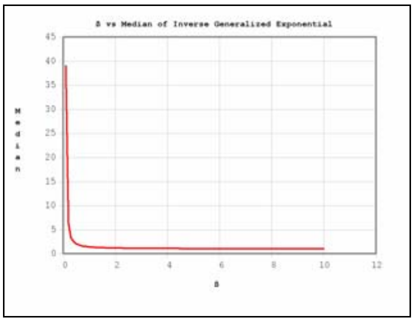

Fig. 3.2 β vs median life/η

3.3. Mode Life

The mode life of the Inverse Generalized Exponential distribution is defined as

$$Mode_{IGE} = t_0 + \frac{\beta}{2\eta} \tag{3.3}$$

The relationship between β and (mode life/η) is shown in Fig 3.3, It is clear that there is no mode life when $\beta = 0$. Mode life/η becomes asymptotically increasing as $\beta \to \infty$. Again β and mode life/η have a positive proportion when β > 0.

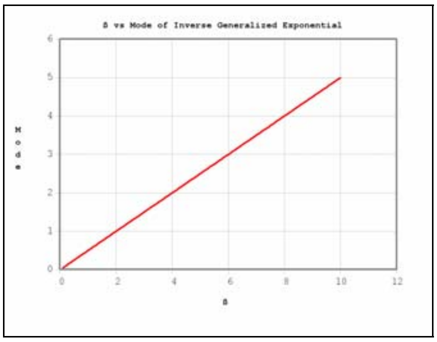

Fig. 3.3 β vs mode life/η

3.4. Variance Life

The variance life of the Inverse Generalized distribution is defined as

$$VAR_{IGE} = \frac{\eta^2}{\beta^2} \tag{3.4}$$

From our calculations it is clear that there is no variance life when $\beta \le 0$. We obtain the maximum value of variance life is 100 for β =0.1. The relationship between β and the variance/η^2 life is shown in Fig 3.4. It is clear that the larger the value of β the smaller the value of variance life/η^2. The relationship between β and the variance/η^2 life shows that it becomes asymptotic to 0 as $\beta \to \infty$. The Inverse Generalized Exponential distribution standard deviation SDIGE is the measure of spread, and can be obtained by taking the square root of the VAR_{IGE}.

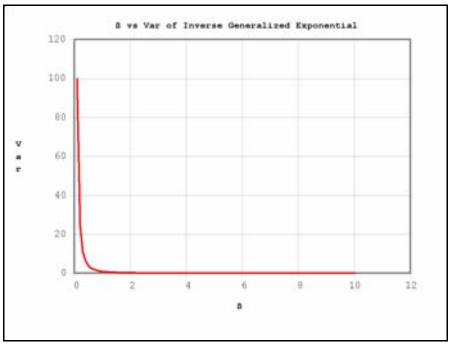

Fig 3.4 β vs variance/η^2

3.5. Coefficient of Variation

The coefficient of variation CVIGE is defined as

$$SD_{IGE}/MTTF_{IGE}\,, \quad CV_{IGE} = 1 \tag{3.5}$$

From my calculation it is clear that there is unit coefficient of variation life. The relationship between β and CVIGE is fixed. The larger or the smaller the value of β has no effect on the value of CVIGE.

3.6. Coefficient of Skewness

The coefficient of skewness CS_{IGE} is defined as $E(t^{-1} - E(t^{-1}))^3 / \left(E(t^{-1} - E(t^{-1}))^2\right)^{\frac{3}{2}}$

$$E(t^{-1} - E(t^{-1}))^3 = \frac{2\eta^3}{\beta^3} \quad \text{(3.6a)}, \qquad E(t^{-1} - E(t^{-1}))^2 = \frac{\eta^2}{\beta^2} \quad \text{(3.6b)}$$

Where $CS_{IGE} = 2$ is the quantity used to measure the skewness of the Inverse Generalized Exponential distribution, here CS_{IGE} > 0 then the PDF of the Inverse Generalized Exponential distribution is skewed to the right when (Mean > Median > Mode). The relationship between β and CS_{IGE} is a fix value. From my calculations it is clear that there is no CS_{IGE} life when $\beta < 0$.

3.7. Coefficient of Kurtosis

The coefficient of kurtosis CK_{IGE} is defined as $E(t^{-1} - E(t^{-1}))^4 / \left(E(t^{-1} - E(t^{-1}))^2\right)^2$

$$E(t^{-1} - E(t^{-1}))^4 = \frac{9\eta^4}{\beta^4} \tag{3.7}$$

Where $CK_{IGE} = 9$, this quantity is used to measure the kurtosis or peaked ness of the distribution. The Inverse Generalized exponential PDF shape is more peaked than the Normal PDF because the value of CK_{IGE} > 3.

These Properties $f_{\mathrm{IGE}}(t), F_{\mathrm{IGE}}(t), R_{\mathrm{IGE}}(t), h_{IGE}(t), H_{\mathrm{IGE}}(t)$, mean, median, mode and variance can be used to measure the life of system or process. It is important to note that CVIGE, CS_{IGE} and CK_{IGE} are independent of t_0 and CVIGE, CS_{IGE} and CK_{IGE} are all have fix values. It is worth mentioning that when η = 1 and t_0 = 0, the distribution is sometimes called the standard Inverse Generalized exponential distribution under the relationship between β and these properties.

4. Summary and Conclusions

In this paper, the behaviors of the probability distribution (PDF), cumulative distribution function (CDF), reliability function (RF), hazard function (HF), cumulative hazard function (CHF) are investigated based on different choice of parameters. We have seen that the Inverse Generalized Exponential distribution is the flexible distribution model that approaches to inverse exponential distribution when its shape parameter becomes one. We have also analyzes the behaviors of the mean, median, mode and variances corresponding their shape parameter.

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