2012 IACSIT Coimbatore Conferences
IPCSIT vol. 28 (2012) © (2012) IACSIT Press, Singapore

Exactly Solvable Quantum Mechanical Potentials Associated with Romanovski Polynomials

Atreyee Biswas

Department of Natural Science and Humanities
West Bengal University of Technology, Salt Lake City, Kolkata-700064

Abstract. In this paper, both time independent and time dependent Schrödinger equation with position dependent mass has been solved in terms of Romanovski polynomial which is not so widespread like other hypergeometric polynomials. For both cases corresponding supersymmetric partner potentials are also found.

Keywords: point canonical transformation, effective mass, Romanovski polynomial, supersymmetry

1. Introduction

Romanovski polynomials were discovered in 1884 by Routh [1] in the form of complexified Jacobi polynomials on the unit circle in the complex plane and were then rediscovered within the context of probability distributions as real polynomials by Romanovski [2]. Romanovski polynomials may be derived as the polynomial solutions of the ordinary differential equation

$$\left(1+x^2\right)\frac{d^2R(x)}{dx^2}+\left(2\beta x+\alpha\right)\frac{dR(x)}{dx}+\lambda R(x)=0 \tag{1}$$

which is a particular subclass of the hypergeometric differential equations [3,4]. Other subclasses give rise to the well known classical orthogonal polynomials of Hermite, Laguerre and Jacobi type [4,5]. In contrast with the latter orthogonal polynomials, Romanovski polynomials are not orthogonal with respect to the weight function $\omega^{(\alpha,\beta)}$ in the interval $\left(-\infty,\infty\right)$. Specifically, if $R_m^{(\alpha,\beta)}(x)$ and $R_n^{(\alpha,\beta)}(x)$, $m\neq n$ are Romanovski polynomials of degree m and n respectively, then [6]

$$\int_{-\infty}^{\infty}\omega^{(\alpha,\beta)}(x)R_m^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(x)dx=0 \tag{2}$$

if and only if $m+n<1-2\beta$.

It is this finite orthogonality relation (giving rise to finite number of bound states) which makes it distinct from other orthogonal polynomials. Recently several problems i) the time independent Schrödinger equation with hyperbolic Scarf and trigonometric Rosen Morse potential [6], ii) Klein Gordon equation with equal vector and scalar potential [7], iii) certain classes of non-central potential problems [8] have been solved in terms of Romanovski polynomials.

On the other hand, there exists a wide variety of physical problems in which an effective mass depending on the position is of utmost relevance, such as effective interactions in nuclear physics [10], carriers and impurities in crystals [11], quantum dots [12], semiconductor heterostructures [13], physics in neutron stars [14] etc. Position-dependent masses [PDM] also hold out to deformation in the quantum canonical commutation relations or curvature of the underlying spaces [15,16]. Furthermore, they also appear in nonlinear oscillators [17] and PT symmetric cubic anharmonic oscillator [18].

In this paper we shall obtain exact solutions of time independent and time dependent PDM Schrödinger equation which are given in terms of Romanovski polynomials. The motivation for doing this comes from the fact that Romanovski polynomials are not so widespread in applications compared to other orthogonal polynomials. In obtaining the potentials whose bound state wave functions are given in terms of Romanovski polynomial, we take recourse to point canonical transformation approach [24] which was recently discussed in the PDM background [25]. It should be mentioned here that in [26], the bound state solution of PDM Schrödinger equation are given in terms of Romanovski polynomials for the shifted harmonic oscillator and Rosen Morse I potentials by a method based on deformed shape invariance. Regarding the time dependent case let us mention that since the time dependent PDM Schrödinger equation takes a nonconventional form, it is not always possible to transfer results from the well known constant mass case immediately. Recently, the point transformation (form preserving) transformation technique [27] used to transform any pair of time dependent real potentials related by a Darboux transformation for time dependent constant mass Schrödinger equation into a pair of time independent potentials related by usual Darboux transformation for the stationary Schrödinger equation has been generalized [28] to position dependent mass case such that for each solvable potential for stationary PDM Schrödinger equation there exists a solvable time dependent potential for time dependent PDM Schrödinger equation. Here we shall use this technique to generate exactly solvable time dependent potentials for PDM Schrödinger equation whose bound state wave functions are given in terms of Romanovski polynomials. We shall also obtain the corresponding supersymmetric partner potentials by using the supersymmetry formalism for time dependent PDM Schrödinger equation [29].

2. PCT approach in Time independent PDM Context

The general Hermitian PDM Hamiltonian, initially proposed by VON ROOS [30] in terms of three ambiguity parameters α,β,γ such that $\alpha+\beta+\gamma=-1$ leads to the time-independent Schrödinger equation

$$\left[-\frac{d}{dx}\frac{1}{M}\frac{d}{dx}+V_{eff}\right]\psi(x)=E\psi(x) \tag{3}$$

where the effective potential

$$V_{eff}(x)=V(x)+\frac{1}{2}(\beta+1)\frac{M'''}{M^2}-\left[\alpha\left(\alpha+\beta+1\right)+\beta+1+\frac{M'^2}{M^3}\right] \tag{4}$$

depends on some mass term M(x).Here prime denotes derivative with respect to x and M(x) is the dimensionless form of the mass $m(x)=m_0M(x)$ and we have set $\hbar=2m_0=1$

We look for a solution of the above equation (3) in the following form

$$\psi(x)=f(x)F[g(x)] \tag{5}$$

where f(x),g(x) are two functions of x to be determined and F(g) satisfies the following second order differential equation:

$$\frac{d^2F}{dg^2}+Q(g)\frac{dF}{dg}+R(g)F=0 \tag{6}$$

Since in this paper we are interested only in bound-state solutions, we shall restrict ourselves to polynomial solutions. Substituting $\psi(x)=f(x)F[g(x)]$ in Eq. (3) and comparing the result with Eq. (6), we get the following relations:

$$f(x)\propto\sqrt{\frac{M}{g'}}\exp\left[\frac{1}{2}\int^{g(x)}Q(u)du\right] \tag{7}$$

$$E - V_{eff}(x) = \frac{g'''}{2Mg'} - \frac{3}{4M}\left(\frac{g''}{g'}\right)^2 + \frac{g'^2}{M}\left(R - \frac{\dot{Q}}{2} - \frac{Q^2}{4}\right) - \frac{M''}{2M^2} + \frac{3M'^2}{4M^3} \tag{8}$$

This is clear that we need to find some functions M(x), g(x) ensuring the presence of a constant term on the right-hand side of Eq. (6) to compensate E on its left-side and giving rise to an effective potential with well-behaved wave functions. In order to obtain a physically acceptable solution the wave function $\psi(x)$ has to satisfy the following two conditions

a) $\int_D |\psi(x)|^2 < \infty$

b) The hermicity of the Hamiltonian in the Hilbert space spanned by the eigen functions of the potential V(x) is ensured by the following extra condition [34]

$$\frac{|\psi(x)|^2}{\sqrt{M(x)}} \to 0$$

at the end points of the interval where V(x) and n(x) are defined. This condition imposes an additional restriction whenever the mass function M(x) vanishes at any one or both the end points of D.

Now, we can choose M(x) in many ways, for example $M(x) = \lambda g'^2(x)$ [31,32], $M(x) = \lambda g'(x), M = \frac{\lambda}{g'(x)}$ [33], λ being a constant. We choose here $M = \lambda g'(x)$ reducing the Eq. (8) to

$$E - V_{eff}(x) = \frac{g'}{\lambda}\left(R - \frac{1}{2}\frac{dQ}{dg} - \frac{Q^2}{4}\right) \tag{9}$$

If we assume F(g) to be Romanovski polynomial, i.e. $F_n(g) \propto R_n^{(a,b)}(g)$ where $a, b \in \mathbb{R}$ and n=0,1,2……., then for this polynomial

$$Q(g) = \frac{2bg + a}{1+g^2} \quad , \qquad R(g) = \frac{-n(2b+n-1)}{1+g^2}$$

With these values of Q(g) and R(g) Eq.(8) becomes

$$E - V_{eff}(x) = \frac{g'}{\lambda}\left[\frac{-n(2b+n-1)}{1+g^2} - \frac{b + \frac{a^2}{4}}{(1+g^2)^2} + \frac{g(a-ab)}{(1+g^2)^2} + \frac{g^2(b-b^2)}{(1+g^2)^2}\right] \tag{10}$$

Here to generate a constant term on the right-hand side of the above equation, we put $\frac{g'}{\lambda(1+g^2)} = C$ where C>0 is a constant. This gives $g = \tan(qx)$ and $M(x) = \sec^2(qx)$ where we have set $q^2 = C$ so that $\lambda = \frac{1}{q}$.

The energy eigen values and the potential $V_{eff}(x)$ are then given by

$$E_n = -nq^2(2b+n-1) + V_0 \qquad n = 0,1,2.... \tag{11}$$

and

$$V_{eff}(x) = q^2\left[\left(2b + \frac{a^2}{4} - b^2\right)\cos^2(qx) + a(b-1)\sin(qx)\cos(qx) + b(b-1)\right] \tag{12}$$

Therefore, from equation (7) and (5) we get the following results:

$$f(x) \propto \{\sec(qx)\}^b \, e^{\frac{aqx}{2}} \tag{13}$$

$$\text{and } \psi_n(x) \propto (\sec(qx))^b \, e^{\frac{aqx}{2}} R_n^{(a,b)}(\tan(qx)) \tag{14}$$

where $a, b \in \mathbb{R}$ and $n = 0,1,2....$

To be physically acceptable $\psi_n(x)$ has to satisfy the two conditions stated earlier. From condition (a), we get $\psi_n(x)$ is square integrable for $b \le -n$, $n = 0,1,2....$ and the domain of definition becomes restricted to $\left(-\frac{\pi}{2q}, \frac{\pi}{2q}\right)$. Condition (b) imposes no extra restriction.

2.1 Supersymmetric Quantum Mechanics

In this section we have found the supersymmetric partner potential of V_{eff} in Eq. (12). We define two operators A and $A^\dagger$ as:

$$A\psi = \frac{1}{\sqrt{M}}\frac{d\psi}{dx} + B\psi \text{ and } A^\dagger\psi = -\frac{d}{dx}\left(\frac{\psi}{\sqrt{M}}\right) + B\psi$$

where $B(x) = -\frac{1}{\sqrt{M}}\frac{\psi_0'}{\psi_0}$ is the superpotential such that the Hamiltonian of equation (3) can be factorized in the following way

$$H_{eff} = A^\dagger A = \left[-\frac{d}{dx}\frac{1}{M(x)}\frac{d}{dx} + V_{eff}\right] \tag{15}$$

It's supersymmetric partner potentials is given by

$$H_{1,eff} = AA^\dagger = \left[-\frac{d}{dx}\frac{1}{M(x)}\frac{d}{dx} + V_{1,eff}\right] \tag{16}$$

Here, V_{eff} and $V_{1,eff}$ are the supersymmetric partner potentials and are given by

$$V_{eff}(x) = B^2 - \left(\frac{B}{\sqrt{M}}\right)' \tag{17}$$

$$V_{1,eff}(x) = V_{eff}(x) + \frac{2B'}{\sqrt{M}} - \left(\frac{1}{\sqrt{M}}\right)\left(\frac{1}{\sqrt{M}}\right)'' \tag{18}$$

The two potentials $V_{eff}(x)$ and $V_{1,eff}(x)$ are said to be shape-invariant if they satisfy the following condition [35, 36],

$$V_{1,eff}(x, a_1) = V_{eff}(x, a_2) + R(a_1) \tag{19}$$

where a_1 is a set of parameters, a_2 is some function of a_1 and $R(a_1)$ is independent of x.

In case of unbroken supersymmetry, the energy spectrum and wave functions of two such shape invariant effective mass potentials are related by [35]

$$E_0^{eff} = 0$$

$$E_n^{1,eff} = E_{n+1}^{eff} \qquad n = 0,1,2....$$

and $$\psi_n^{(1,eff)} = \frac{A\psi_{n+1}^{eff}}{\sqrt{E_{n+1}^{eff}}}$$

$$\psi_{n+1}^{eff} = \frac{A^{?}\psi_n^{,eff}}{\sqrt{E_n^{1,eff}}}$$

Now, using the specific choice for B(x) stated above and equation (14) we have

$$B(x) = -q\left(b\sin(qx) + \frac{a}{2}\cos(qx)\right) \tag{20}$$

Also, from Eq. (12), (19) and (20) we obtain the supersymmetric partner potential $V_{1,eff}$:

$$V_{1,eff} = q^2\left[\left(\frac{a^2}{4} - b^2 + 1\right)\cos^2(qx) + ab\sin(qx)\cos(qx) + b(b-1)\right] + V_0 \tag{21}$$

We observe that the potential $V_{eff}(x)$ in Eq. (12) and its supersymmetric partner potential $V_{1,eff}(x)$ in Eq. (21) satisfy the following equation:

$$V_{1,eff}(x,b) = V_{eff}(x,b+1) + V_0 - 2b \tag{22}$$

showing that these potentials are shape-invariant.

Ignoring the normalization constants we can write the eigen functions of H_1 as

$$\psi_n^{1,eff} \propto \frac{(2b+n)(n+1)\left[\sec(qx)\right]^{b+1} e^{\frac{aqx}{2}}}{\sqrt{V_0 - q^2(n+1)(2b+n)}} R_n^{a,b+1}(\tan(qx)) \qquad n = 0,1,2.... \tag{23}$$

Since the above wave functions involve Romanovski polynomials, here also only a finite number of states are normalizable.

3. Generation of Exactly Solvable Potential for Effective Mass Time Dependent Schrödinger Equation

We consider the effective mass TDSE

$$i\psi_t + \tfrac{1}{2m}\psi_{xx} - \frac{m}{2m^2}\psi_x - V\psi = 0 \tag{24}$$

where $m = m(x,t)$ is real-valued and positive mass. V = V (x, t) stands for the potential and $\psi = \psi(x,t)$ is the solution.

In Appendix we have found the solution of a constant mass Stationary Schrödinger equation (SE) in terms of Romanovski polynomial and also have found the corresponding potential. The solution of that constant mass stationary SE(46) [Appendix] and the Effective mass Time Dependent Schrödinger equation (TDSE) (24) are related to each other via the following point canonical transformation:

$$\psi(x,t) = \exp\left[h(x,t) - iE\nu(t)\right]\phi\left(u(x,t)\right) \tag{25}$$

Here, $\phi(u)$ is solution of the constant mass stationary SE(56) in transformed co-ordinate u = u(x, t) and E is given by Eq. (53).The functions $h, \nu(t)$ and u read as follows:

$$\begin{aligned} h &= \int\left[\tfrac{m_x}{4m} - i\left(\tfrac{\sqrt{m}}{2}I_2 + \tfrac{2A\sqrt{m}}{M}I_1 + B\sqrt{\tfrac{m}{M}}\right)dx\right] \\ u(x,t) &= \sqrt{\tfrac{1}{M}}I_3 I_1 + \tfrac{1}{M}\int BI_3 dt \\ \nu &= \int I_3^2 dt \end{aligned} \tag{26}$$

with arbitrary real valued A(t), B(t) and d(t). Here the following abbreviations are used:

$$I_1 = \int\sqrt{m}dx, \qquad I_2 = \int\tfrac{m_t}{\sqrt{m}}dx, \qquad I_3 = \exp\left[\tfrac{2}{M}\int Adt\right] \tag{27}$$

Substituting Eq. (25) into the Eq. (24) and demanding that the resulting equation takes the form of constant mass stationary SE(54) in the transformed co-ordinate u gives the potential V(x, t) present in Eq. (24) as

$$V(x,t)=U(u)I_3^2-\frac{2AB}{M^{3/2}}I_1-2\left(\frac{A}{M}I_1\right)^2-\frac{B}{2\sqrt{M}}I_2-\frac{A}{M}I_1I_2-\frac{1}{8}I_2{}^2-\frac{7m_x}{32m^3}+\frac{m_{xx}}{8m^2}+$$
$$\int\left[\left(\tfrac{A\sqrt{m}}{M}+\tfrac{m_t}{4\sqrt{m}}\right)I_2+\tfrac{\sqrt{m}}{2}\left(I_2\right)_t+\left(\tfrac{2A\sqrt{m}}{M}+\tfrac{Am_t}{M\sqrt{m}}\right)I_1+\tfrac{B\sqrt{m}}{M}+\tfrac{Bm_t}{2\sqrt{mM}}\right]dx+i\left(d'-\tfrac{A}{M}\right)-\frac{B^2}{2M} \tag{28}$$

Here U(u) is the potential given in Eq. (62) [in Appendix] for constant mass Stationary SE in new co-ordinate u.

The above potential is a complex-valued function. Since, introduction of a complex potential in general destroys the hermicity of the Hamiltonian and since vast majority of physical interaction is described by real-valued potential, we set up a condition $d'=\dfrac{A}{M}$ in order to make the potential a real valued one.

Also, this is important to say that it has been shown in Ref. [28] that the transformation (24) preserves L^2 -normalizability, i.e. if the solution ϕ of Eq. (46) is -normalizable in a domain $u(D), D\in\mathbb{R}$ then the solution of equation (24) is L^2 -normalizable on D.

Our task is now to find the potential V(x, t) given in Eq. (28). For this the mass function m(x, t) has to be specified. Below we shall consider two physically relevant mass functions.

Example: $m(x,t)=\dfrac{\alpha}{2}\cosh^2 x$ where $\alpha(t)$ is an arbitrary real and positive function. This form of position dependent mass is considered in graded alloys. We set here for simplicity $A=-\dfrac{M\alpha'}{4\alpha}$. On substituting the above settings into (26)-(27), we get

$$I_1=\sqrt{\frac{\alpha}{2}}\sinh x,\qquad I_2=\frac{\alpha'}{\sqrt{2\alpha}}\sinh x,\qquad I_3=\frac{1}{\sqrt{\alpha}}$$
$$\nu=\int\tfrac{1}{\alpha}dt,\qquad u=\frac{1}{\sqrt{2M}}\sinh x,\qquad h=\tfrac{1}{2}\log\left|\cosh x\right|+d(t) \tag{29}$$

Therefore the potential in (28) reads

$$V(x,t)=\frac{q^2}{2M\alpha}\left[\left(\frac{3}{4}-2b-\frac{a^2}{4}+b^2\right)\tanh^2\left(\frac{q\sinh x}{\sqrt{2M}}\right)+a(b-1)\tanh\left(\frac{q\sinh x}{\sqrt{2M}}\right)\sec h\left(\frac{q\sinh x}{\sqrt{2M}}\right)+b+\frac{a^2}{4}-\frac{1}{2}\right]$$
$$+i\left(d'+\frac{\alpha'}{4\alpha}\right)-\frac{\left(3-5\sec h^2x\right)\sec h^2x}{4\alpha} \tag{30}$$

The solution of the time-dependent position dependent mass Schrödinger equation (24) in the present case can be obtained with the help of Eq. (25), (29) and (56) [Appendix] as

$$\psi_n(x,t)\propto\sqrt{\cosh x}\left[\cosh\left(\tfrac{q\sinh x}{\sqrt{2M}}\right)\right]^{b-1/2}\exp\left[-i\frac{nq^2}{2M}(2b+n-1)\int\tfrac{1}{\alpha}dt+\frac{a}{2}\tan^{-1}\sinh\left(\tfrac{q\sinh x}{\sqrt{2M}}\right)\right]$$
$$\times R_n^{(a,b)}\left[\sinh\left(\tfrac{q\sinh x}{\sqrt{2M}}\right)\right] \tag{31}$$

Next, we concentrate ourselves in finding out the supersymmetric partner potential of the time-dependent position dependent potential V(x, t) for each of the cases.
The Hamiltonian corresponding to the TDSE (24) is

$$H_0=\frac{1}{2}\left(p\frac{1}{m}p\right)+V \tag{32}$$

Let us consider another Hamiltonian with same kinetic energy term but with different potential $V_1(x,t)$:

$$H_1=\frac{1}{2}\left(p\frac{1}{m}p\right)+V_1 \tag{33}$$

Let ψ and ϕ be solutions of the effective mass time-dependent Schrödinger equations (TDSE) associated with the Hamiltonians H_0 and H_1 respectively. Now, combining the TDSE-s corresponding to these Hamiltonians (35) and (36) in one single matrix TDSE in the form

$$\left[\begin{pmatrix} i\partial_t & 0 \\ 0 & i\partial_t \end{pmatrix} - \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix}\right]\begin{pmatrix} \psi \\ \varphi \end{pmatrix} = 0 \tag{34}$$

On defining $H = diag\left(H_0, H_1\right)$ and $\Psi = \left(\psi, \phi\right)^T$ the above matrix TDSE can be written as

$$\left[i\partial_t - H\right]\Psi = 0 \tag{35}$$

Two supercharge operators Q and $Q^\dagger$ are defined as follows

$$Q = \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix} \text{ and } Q^\dagger = \begin{pmatrix} 0 & D^\dagger \\ 0 & 0 \end{pmatrix} \tag{36}$$

where D and $D^\dagger$ are given by $D = \sqrt{\frac{1}{2}} m^{-1/4}\partial_x m^{-1/4} + W - \frac{i}{4\sqrt{2}}\int \frac{m_t}{\sqrt{m}} dx$ (37)

$$D^{?} = -\sqrt{\frac{1}{2}} m^{-/4}\partial_x m^{-1/4} + W - \frac{i}{4\sqrt{2}}\int \frac{m_t}{\sqrt{m}} dx \tag{38}$$

Here $W = W(x,t)$ is an arbitrary function called the superpotential. $Q, Q^\dagger$ and H satisfy the following algebras

$$\begin{aligned} \left[Q, Q\right]_+ &= \left[Q^\dagger, Q^\dagger\right]_+ = 0 \\ \left[Q, Q^\dagger\right]_+ &= \left[Q\ , Q\right]_+ = 0 \\ \left[Q, i\partial_t - H\right] &= \left[i\partial_t - H, Q\right] = 0 \end{aligned} \tag{39}$$

As for the second line in (39), the anticommutator conditions imply that the Hamiltonians H_0 and H_1 in the matrix H must factorise as follows:

$$H_0 = D^\dagger D \text{ and } H_1 = DD^\dagger$$

On employing the explicit forms (37), (38) of the operators D and D† respectively in the above relations we find the following time-dependent supersymmetric partner potentials $V(x,t)$ and $V_1(x,t)$ [29] as

$$V = W^2 - \sqrt{\frac{1}{2m}} W_x - \left(\frac{i}{2\sqrt{2}}\int \frac{m_t}{\sqrt{m}} dx\right) W + \frac{m_{xx}}{8m^2} - \frac{7m_x^2}{32m^3} - \frac{1}{32}\left(\int \frac{m_t}{\sqrt{m}} dx\right)^2 + i\frac{m_t}{8m} \tag{40}$$

$$V_1 = W^2 + \sqrt{\frac{1}{2m}} W_x - \left(\frac{i}{2\sqrt{2}}\int \frac{m_t}{\sqrt{m}} dx\right) W + \frac{m_{xx}}{8m^2} - \frac{7m_x^2}{32m^3} - \frac{1}{32}\left(\int \frac{m_t}{\sqrt{m}} dx\right)^2 - i\frac{m_t}{8m} \tag{41}$$

V_1 in terms of V is then can be written as $V_1 = V + \sqrt{\frac{2}{m}} W_x - \frac{i}{4m} m_t$ (42)

In Ref. [29] this has been shown that Supersymmetry formalism and Darboux transformation are equivalent. Here, W is taken in the form:

$$W = -\sqrt{\frac{1}{2m}}\frac{v_x}{v} + \frac{m_x}{4\sqrt{2}m^{3/2}} + \frac{i}{4\sqrt{2}}\int \frac{m_t}{\sqrt{m}} dx \tag{43}$$

where $v = v(x,t)$ solves the effective mass TDSE: $\left[i\partial_t - H_0\right]\Psi=0$

On taking $W = W(x,t)$ in the above form the D operator in (40) becomes a Darboux operator establishing the equivalence of supersymmetry formalism for effective mass with the effective mass Darboux transformation.

In this paper, we take $v(x,t) = \psi_0(x,t)$, ψ_0 being the ground state wave function for the TDSE (24)

Let us now find out the supersymmetric partner potential for the potential given in the equation (30).

Example: Here $m(x,t) = \frac{\alpha}{2}\cosh^2 x$, $\alpha = \alpha(t)$ being an arbitrary positive real function. From Eq. (31), the ground-state wave function in this case we obtain as:

$$\psi_0(x,t) \propto \sqrt{\cosh x}\left(\cosh\left(\frac{q\sinh x}{\sqrt{2M}}\right)\right)^{b-\frac{1}{2}} \exp\left[\frac{a}{2}\tan^{-1}\sinh\left(\frac{q\sinh x}{\sqrt{2M}}\right)\right] \tag{44}$$

Using Eq. (43) and (44) we obtain the supersymmetric partner potential for the potential given in Eq. (30) as:

$$V_1 = V + \frac{q^2}{M\alpha}\sec h\left(\frac{q\sinh x}{\sqrt{2M}}\right)\left[\left(b-\frac{1}{2}\right)\tan h\left(\frac{q\sinh x}{\sqrt{2M}}\right) - \frac{a}{2}\sec h\left(\frac{q\sinh x}{\sqrt{2M}}\right)\right]$$

$$= \frac{q^2}{2M\alpha}\left[\left(b^2 - \frac{a^2}{4} - \frac{1}{4}\right)\tanh^2(qu) + ab\sec h(qu)\tanh(qu) + \frac{a^2}{2} - b + \frac{1}{2}\right] + \frac{\sec h^2(x)\left(5\sec h^2(x) - 3\right)}{4\alpha}$$

$$\text{where } u = \frac{1}{\sqrt{2M}}\sinh x \tag{45}$$

4. Conclusion

We obtained exactly solvable potentials for both time-dependent and time independent effective mass Schrödinger equations whose bound state solutions are given in terms of Romanovski polynomial. We have also obtained their super symmetric partner potentials. They all depend on time dependent parameter α strongly. Using the same approach we can solve time-dependent effective mass Schrödinger equations whose bound state solutions are given in terms of other hypergeometric polynomials.

5. Acknowledgement

It is a pleasure to thank Dr. Barnana Roy for her valuable suggestions and guidance.

6. References

[1] E. J. Routh. *Proc. London. Math. Soc.* 16 (1884) 245

[2] V. Romanovski. *C. R. Acad. Sci. Paris*. 188 (1929) 1023

[3] A. F. Nikiforov and V. B. Uvarov. *Special Functions of Mathematical Physics*. (Birkh¨auser Verlag, Basel.1988)

[4] P. A. Lesky, Z. Angew. Math. Mech. 76 (1996) 181

[5] H. J. Weber. *C. E. J. Math*. (2007)

[6] C. B. Compean and M. Kirchbach. *J. Phys. A* 39 (2006) 547

[7] D. E. Alvarez-Castillo and M. Kirchbach. *Rev.Mex.Fis*. E53 (2007) 143

[8] W. C. Qiang. *Chinese. Phys*. 12 (2003) 136

[9] W. C. Qiang and S. H. Dong. *Phys. Script*. 72 (2005) 127

[10] P. Ring and P. Schuck. *The nuclear many-body problem, Texts and Monographs in Physics* (Springer-Verlag, New York, Berlin 1980)

[11] J. M. Luttinger and W. Kohn. *Phys. rev*. 97 (1955) 869
[12] P. Harrison. *Quantum wells, wires and dots, Wiley* (Chichester, 1999)
[13] F. Arias de Saavedra et al, *Phys. Rev*. B 50 (1994) 4248
[14] N. Channel. *Nucl. Phys.* A 773 (2006) 263
[15] C. Quesne and V. M. Tkachuk. *J. Phys.* A 37 (2004) 4267
[16] I. Infeld and A. Schild. *Phys. Rev*. 67 (1945) 121
[17] J. F. Carin˜ena et al. *Ann. Phys*. 322 (2007) 434
[18] A. Mostafazadeh, *J. Phys*. A 40, 6557 (2005) [Erratum ibid. 38, 8185 (2005)]
[19] B. Midya and B. Roy, *J. Phys*.A 42 (2009) 285301
[20] A. Schulze-Halberg. *Int. J. Mod. Phys*. A 22 (2007) 3293
[21] A. A. Suzko and A. Schulze-Halberg. *Phys. Letts*. A 372 (2008) 5865
[22] A. Schulze-Halberg et al. *J. Phys*. A 42 (2009) 115211
[23] A. Schulze-Halberg. *Cent. Eur. J.Phys*. 3 (2005) 591
[24] A. Bhattacharjie and E.C.G. Sudarshan. *Nuovo Cimento* 25 (1962) 262
[25] B. Bagchi et al. *Europhys. Letts*. 72 (2005) 55
[26] C. Quesne. *SIGMA* 5 (2009) 046
[27] F. Finkel et al. *Jour. Math. Phys*. 40 (1999) 3268
[28] A. Schulze-Halberg. *Res. Letts. Phys*. (2008) doi:10.1155/2008/589269
[29] A. Schulze-Halberg. *Cent .Euro. Jour. Phys*. 6 (2008) 153
[30] O. Von Roos. *Phys. Rev. B* 27(1983)7547
[31] B. Bagchi, P. Gorain, C. Quesne and R. Roychoudhury. *Mod. Phys. Lett*. A 19 (2004) 2765-2775
[32] A. D. Alhaidari. *Phys. Rev*. A 66 (2002) 042116
[33] B. Bagchi, P. Gorain, C. Quesne and R. Roychoudhury. *Europhys. Lett*. 72 (2005) 155-161
[34] B. Bagchi, A. Banerjee, C. Quesne and V. M. Tkachuk. *J. Phys. Rev*. A 38 (2005) 2929
[35] A. R. Plastino,A. Rego,M. Casas,F. Gracias and A. Plastino. *Phys.Rev.A* 60 (1999) 4398
[36] F. Cooper, A. Khare and U. Sukhatme. *Phys. Rep*. 251 (1995)267

7. Appendix

We consider the stationary SE $\frac{1}{2M}\frac{d^2\phi}{dx^2}+\left(E-U\right)\phi=0$ (46)

where the mass M is a positive constant, $U=U(x)$ denotes the potential with constant energy E and $\phi=\phi(x)$ is the solution. Let us look for solution of the equation (6) of the form

$$\phi(x)=f(x)F(g(x)) \tag{47}$$

where $F(g)$ is a special function which satisfies a second order differential equation:

$$\frac{d^2F}{dg^2}+Q(g)\frac{dF}{dg}+R(g)F=0 \tag{48}$$

Substituting $\phi(x)=f(x)F(g(x))$ in Eq. (1) we get the following formulae:

$$E-U(x)=\frac{g'''}{4Mg'}-\frac{3}{8M}\left(\frac{g''}{g'}\right)^2+\frac{g'^2}{2M}\left(R-\frac{\dot{Q}}{2}-\frac{Q^2}{4}\right) \tag{49}$$

$$f(x)\propto\frac{1}{\sqrt{g}}\exp\left[\frac{1}{2}\int^{g(x)}Q(u)du\right] \tag{50}$$

Now, we assume $F(g)$ to be Romanovski polynomial. So, $F_n(g) \propto R_n^{(a,b)}(g), n = 0,1,2........$ and $F_n(g)$ satisfies the Eq. (1). Therefore, we obtain $Q(g) = \dfrac{2bg+a}{1+g^2}$ and $R(g) = \dfrac{-n(2b+n-1)}{1+g^2}$

So, $$R - \frac{\dot{Q}}{2} - \frac{Q^2}{4} = \frac{-n\left(2b+n-1\right)}{1+g^2} - \frac{b+\dfrac{a^2}{4}}{\left(1+g^2\right)^2} + \frac{g\left(a-ab\right)}{\left(1+g^2\right)^2} + \frac{g^2\left(b-b^2\right)}{\left(1+g^2\right)^2} \tag{51}$$

A constant term can be generated on the right side of Eq. (4) if we assume $\dfrac{g'^2}{2M\left(1+g^2\right)} = C$ (constant). We take C as positive in order to get increasing energy eigen values for successive values of n. The solution of the above first-order differential equation for $g(x)$ reads

$$g(x) = \sinh(qx) \text{ where } q = \sqrt{2MC} > 0 \tag{52}$$

From equations (7), (9), (10), (11) and (12), we get the energy eigen values, potential and also the bound-state solutions.

$$E_n = \frac{nq^2}{2M}\left(2b+n-1\right),\ n = 0,1,2........ \tag{53}$$

$$U(x) = \frac{q^2}{2M}\left[\left(\frac{3}{4} - 2b - \frac{a^2}{4} + b^2\right)\tanh^2\left(qx\right) + a(b-1)\tanh\left(qx\right)\sec h\left(qx\right) + b + \frac{a^2}{4} - \frac{1}{2}\right] \tag{54}$$

$$f(x) \propto \left[\cosh(qx)\right]^{b-\frac{1}{2}} \exp\left[\frac{a}{2}\tan^{-1}\sinh(qx)\right] \tag{55}$$

$$\phi_n(x) \propto \left[\cosh(qx)\right]^{b-\frac{1}{2}} \exp\left[\frac{a}{2}\tan^{-1}\sinh(qx)\right] R_n^{(a,b)}\left(\sinh(qx)\right) \tag{56}$$

From properties of Romanovski polynomial this is known to us that if $R_m^{(a,b)}(x)$ and $R_n^{(a,b)}(x), m \neq n$, are Romanovski polynomials of degree m and n respectively, then

$$\int_{-\infty}^{\infty} \omega^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(x)dx = 0 \tag{57}$$

iff $m+n < 1-2\beta$ and for the family of polynomials $\Re^{(\alpha,\beta)}$ only the polynomials $R_n^{(\alpha,\beta)}(x)$ with $n < -\beta$ are normalizable i.e. the integral $\int_{-\infty}^{\infty} \omega^{(\alpha,\beta)}(x)(R_m^{(\alpha,\beta)}(x))^2 dx$ converges , $\omega^{(\alpha,\beta)}$ being the weight function given by $\omega^{(\alpha,\beta)} = \left(1+x^2\right)^{\beta-1}\exp\left[\alpha\cot^{-1}x\right]$.

So, for the family $\Re^{(\alpha,\beta)}$ only the polynomials in a finite subset are normalizable and only a finite number of couples are orthogonal. Taking account into above properties of Romanovski polynomials this can be shown that the polynomials $\phi_m(x)$ and $\phi_n(x)$ (given by Eq.(61)), $m \neq n$ are orthogonal iff $m+n < 1-2\beta$ and only the polynomials $\phi_n(x)$ with $b\langle\frac{1}{2} - n, n = 0,1,2..........$ are normalizable in the whole real line.