An Improved Algorithm and It’s Application to Sinusoid Wave Frequency Estimation

Xiaohong Huang and Lin Zhang

1College of Information Engineering, Hebei United University, Tangshan, Hebei, China
2College of Electrical Engineering, Hebei United University, Tangshan, Hebei, China
1tshxh@163.com and 2Zhanglin565650734@126.com

Abstract. An improved algorithm based on Kay’s estimator and its application to Sinusoid Wave frequency estimation are investigated. Firstly a new method of spectrum analysis is introduced, which has excellent performance in suppressing spectral leakage and the property of ‘phase invariant’, then a hybrid All Phase Kay (ApIkay) algorithm is proposed, which merges Kay’s estimator and phase unwrapping. The improved algorithm is applied to the frequency estimation of a sinusoid, the frequency performance is better than Kay and Ikay. When SNR(<7 dB), The simulation results show that the mean square error of the new frequency estimator is improved 4dB than Kay, 1 dB than Ikay. When SNR(>7 dB), the performance of ApIkay obtains CRLB quickly, and the performance is stable in the whole frequency range.

Keywords: frequency estimation, Kay, phase unwrapping, sinusoid, CRLB

1. Introduction

Estimating the frequency of a single sinusoid corrupted by additive, white, Gaussian noise (AWGN) is an important and classical problem in communications, radar and sonar signal processing. Maximum likelihood (ML) frequency estimators in frequency-domain were studied by Rife and Boorstyn in [1], which has large complexity. The time-domain estimators in [2-8] are derived from the ML principle. Tretter proposed unwrapping the signal phase and performing linear regression to obtain a frequency estimate [2], but can only work well at high signal-to-noise ratio (SNR). Kay addressed the phase unwrapping problem by only considering the phase differences and presented a simple frequency estimation algorithm, namely, Kay’s estimator [3], which can approach the Cramer-Rao lower bound (CRLB) at high SNR, but this estimation method has obvious threshold in reality application and has relation with the frequency. The performance becomes bad when frequency is close to 0, half of sample frequency, sample frequency. Ikay in [9] improved Kay’s estimation performance. In this paper, we improve the Ikay further, ApIkay is proposed based on Kay’s estimator and a new method of spectrum analysis and phase unwrapping, which has better performance that the MSE of frequency estimation improves about 4dB than Kay’s and 1 dB than Ikay in the low SNR(<7 dB), and close to CRLB when (>7 dB) quickly.

2. A New FFT Spectrum Analysis

A novel algorithm of spectrum estimation is put forward in the literature [10], named ApFFT, which improves the data truncating way of traditional DFT spectrum analysis and reduces the leakage greatly. The block diagram is shown in the bottom of the figure 1. The 2N-1 order window function is the convolution of two same symmetric N order window.
First, we deduce the amplitude of a signal consisting of a single frequency \( f_0 \). If the signal with single frequency is
\[
\sin(2\pi f_0 t) = e^{j2\pi f_0 t},
\]
where \( f_0 \) is the signal frequency, \( f_s \) is the sample frequency.

To one sample point \( x(N) \) in the time sequence, there are \( N \) vectors of \( N \) dimension including this sample point:
\[
\begin{align*}
X_{\phi_i} &= [ x(0) \ x(1) \ ... \ x(N-1) ]^T, \\
X_i &= [ x(0) \ x(1) \ ... \ x(N-1) ]^T, \\
X_{\phi_i} &= [ x(0) \ x(1) \ ... \ x(N-1) ]^T.
\end{align*}
\]

Cycle shift every vector, shift the sample \( x(N) \) to the first position of the sequence and get the other \( N \) vectors of \( N \) dimension:
\[
\begin{align*}
X_{\phi_i} &= [ x(0) \ x(1) \ ... \ x(N-1) ]^T, \\
X_i &= [ x(0) \ x(1) \ ... \ x(N-1) ]^T, \\
X_{\phi_i} &= [ x(0) \ x(1) \ ... \ x(N-1) ]^T.
\end{align*}
\]

We can get all phase data vector by adding \( N \) vectors aiming at \( x(N) \)
\[
X_{\phi} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi n/N}
\]

According to the shift property of discrete Fourier transform, there has clear relationship between the \( X_i(k) \) and \( X_{\phi}(k) \), where \( X_i(k) \) is the discrete Fourier transform of \( X_i(n) \) and \( X_{\phi}(k) \) is the discrete Fourier transform of \( X(n) \).

\[
X_{\phi}(k) = X_i(k) \frac{e^{-j2\pi k/N}}{N}
\]

ApFFT is made up of the sum of \( X_i(k) \) , so:
\[
X_{\phi}(k) = \frac{1}{N} \sum_{i=0}^{N-1} X_i(k) \frac{e^{-j2\pi k/N}}{N}
\]

According to (2), the amplitude of all phase spectrums is as follows
\[
A(k) = \left| \frac{\sin \pi N (f_0) - k/N}{\sin \pi (f_0) - k/N} \right|, \quad \text{it is the square of}
\]

Another important character of ApFFT spectrum analysis is that its phase is constant and isn’t influenced by the frequency shift, so the phase needn’t to be corrected. That means the real phase of signal can be obtained by ApFFT spectrum analysis when the signal is non-integer truncated. The measured phase value and the real phase value have less error.

Take the signal \( \cos(1.2 \times 2\pi/6t + 100\pi/180) \) as an example to search the reason of so little phase error about ApFFT spectrum analysis. We can get 11 samples:

-0.1736 -0.9903 -0.4384 0.7193 0.8829 -0.1736 -0.9903 -0.4384 0.7193 0.8829 -0.1736

Figure 1: Figure 1 the diagram of ApFFT spectrum analysis
The input signal of all phase is made up of 6 groups of N=6 samples. The first group consists of the last 6 samples among all 11 samples, the second group consists of another 6 samples which left shift 1 value, but -0.1736 should right shift to the first position, other groups could be get as the same way.

<table>
<thead>
<tr>
<th>-0.1736</th>
<th>-0.9903</th>
<th>-0.4384</th>
<th>0.7193</th>
<th>0.8829</th>
<th>-0.1736</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1736</td>
<td>-0.9903</td>
<td>-0.4384</td>
<td>0.7193</td>
<td>0.8829</td>
<td>-0.1736</td>
</tr>
<tr>
<td>-0.1736</td>
<td>-0.9903</td>
<td>-0.4384</td>
<td>0.7193</td>
<td>0.8829</td>
<td></td>
</tr>
<tr>
<td>-0.1736</td>
<td>-0.9903</td>
<td>-0.4384</td>
<td>0.7193</td>
<td>0.8829</td>
<td></td>
</tr>
<tr>
<td>-0.1736</td>
<td>-0.9903</td>
<td>-0.4384</td>
<td>0.7193</td>
<td>0.8829</td>
<td></td>
</tr>
</tbody>
</table>

The phases of the samples of the 6 group of N=6 signals are:

\[
\begin{align*}
180.0000 & \quad 132.5107 & -25.7699 & 0 & -132.5107 & 180.0000 \\
0 & 112.9053 & 51.7298 & 180.0000 & 308.2702 & 247.0947 \\
0 & 112.4978 & 53.4788 & 180.0000 & 306.5212 & 247.5022 \\
180.0000 & 88.4598 & 138.4375 & 0 & -88.4598 & \\
180.0000 & 83.6275 & 161.4769 & 0 & -83.6275 & \\
180.0000 & 72.5107 & 214.2301 & 180.0000 & 259.9996 & 259.9931
\end{align*}
\]

Because the frequency is 1.2, we should observe the second phase in every group. Three are bigger and three are smaller than the real phase (100) during 6 phases. Input data of ApFFT is the average of the above 6 groups’ signals, phases are counteract each other, which make phase difference zero. So the phase got by ApFFT is the signal real phase. The result of experiment shows that when the signal is inter-period sampled, the phase got by ApFFT is perfect. In this case, the phase of signal got by ApFFT with window (kaiser(N,9.5) convolute Kaiser(N,9.5) ) is as follows:

\[
\begin{align*}
180.0000 & \quad 100.0069 & 100.0004 & 180.0000 & 259.9996 & 259.9931
\end{align*}
\]

In this case, the real phase is 100, the measured phase is 100.0069, so the error is only 0.69%, and we can think them very similar. According to the above example, we can get the conclusion that all phase has perfect phase analysis property, especially when the signal is non-inter-period sampled, the phase analyzed by this method is almost the real value; while the phase analyzed by traditional method is deflect from the real value.

So a method of ApFFT phase difference is proposed in [10], which can estimate signal’s parameters with less error, but has not good result in the low SNR.

3. Improved Algorithm based ApFFT and Kay

A mono-component sinusoid contaminated by AWGN can be modeled as:

\[
\begin{align*}
r(n) &= s(n) + w(n) = A \exp(\mathbb{j}\phi(n)) + w(n), n = 1, \ldots, N \\
\phi(n) &= 2\pi f_c n T + \theta_c
\end{align*}
\]

Where A is the amplitude, \( f_c \) and \( \theta_c \) are frequency and initial phase, \( T \) is sample cycle.

We can get phase from (3) from the received signal:

\[
\phi(n) = \arctan \left[ \frac{\text{Im}[r(n)]}{\text{Re}[r(n)]} \right] \tag{5}
\]

But this is not the true phase, the true phase \( \theta_i(n) \) is obtained through phase unwrapping on the measured phase \( \phi(n) \).

In the condition of noise, the relation between true phase and measured phase is as (6),where \( k_i (k_i \in \mathbb{Z}) \) is the number of period of the nth sample.

\[
\phi(n) = \phi_i(n) - k_i 2\pi
\]

The phase difference of adjacent samples \( \Delta \phi \) is:

\[
\Delta \phi(n) = \Delta \phi_i(n) - (k_i - k_{i-1}) 2\pi \tag{7}
\]
(vii) In order to simplify, we named the method of this paper ApIKay, under the low SNR samples is $2N - 1 = 63$, estimate frequencies of this two sinusoidal waves by Kay’s estimator and Ikay and example, frequencies are 20MHz and 40MHz separately, sampling frequency is $f_s = 100$MHz and the length of 4.

(iv) Compute MSE of this frequency points, the result is shown in Table 2. From Table 2, we can see that

The better the performance of phase unwrapping; the closer to True the phase difference of adjacent samples. By using this performance, an improved algorithm based on ApFFT and Kay is proposed, the step of this algorithm is as follows:

a) Obtain $R(k)$ by performing Fourier transform on $r(n)$, then estimate the center frequency $\tilde{f}_c$ of $r(n)$, compute the shift value $f_0 = f_s - \tilde{f}_c$, where $f_s$ is the sampling frequency.

(i) Take $2N - 1$ samples and shift to center frequency, another signal can be obtained:

$$z(n) = r(n) \ast \exp(j2\pi f_c nT),$$

where $2N - 1$ is the samples length of $r(n)$.

(ii) Perform ApFFT on $z(n)$ in order to get phases $\phi_z(n)$, the length of which is N.

(iii) Unwrap $\phi_z(n)$ by using formula (8) to get the true phase $\phi^{\wedge}_z(n)$.

(iv) Get Phase difference $\Delta \phi(n)$ through two slices of phase data with length N-1, which is shift only one point.

(v) Get the unbiased estimator of frequency $\hat{f} = \sum_{n=0}^{N-1} h(n)\Delta \phi(n)$, where $h(n) = \frac{3}{2}N^{-1} \left[1 - \left(\frac{n - \frac{N}{2} - 1}{N} \right)^2\right]$.

(vi) Get the true frequency of $r(n)$: $f = \hat{f} - f_0$.

4. Prepare Your Paper Before Styling

Do some simulations by Matlab to test the improved algorithm. Take two groups of sinusoid wave as example, frequencies are 20MHz and 40MHz separately, sampling frequency is $f_s = 100$MHz and the length of samples is $2^4 N - 1 = 63$, estimate frequencies of this two sinusoid waves by Kay’s estimator and Ikay and ApIkay in order to simplify, we named the method of this paper ApIkay, under the low SNR condition (SNR < 7dB), do Monte-Carlo simulations to get mean square error. The number of simulation runs is set to 1000 for each case; the result is shown in Table 1 and 2.

| TABLE I. FREQUENCY ESTIMATION COMPARISON ($f_c = 20$MHz, UNIT: kHz) |
|--------------------------|---------|-------|-------|-------|-------|-------|
|                         | 0dB     | 1 dB  | 2 dB  | 3 dB  | 4 dB  | 5 dB  | 6 dB  |
| Kay                     | 4331.4  | 3925.4| 3281.8| 2611.8| 1957.5| 1269.3| 814.36|
| Ikay                    | 2249.1  | 1795.4| 1433.5| 1038.2| 685.38| 397.41| 371.65|
| ApIkay                 | 1757.8  | 1456.9| 998.67| 773.66| 488.85| 324.6 | 291.44|
| CRLB                   | 215.4   | 191.9 | 171.1 | 152.5 | 135.9 | 121.1 | 107.9 |

| TABLE II. FREQUENCY ESTIMATION COMPARISON ($f_c = 40$MHz, UNIT: kHz) |
|--------------------------|---------|-------|-------|-------|-------|-------|
|                         | 0dB     | 1 dB  | 2 dB  | 3 dB  | 4 dB  | 5 dB  | 6 dB  |
| Kay                     | 6478.6  | 6353.4| 6189.3| 6171.0| 5997.1| 5680.0| 5482.3|
| Ikay                    | 2291.4  | 1852.5| 1395.2| 1043.6| 637.6 | 436.59| 313.06|
| ApIkay                 | 1757.8  | 1380.9| 1040.4| 789.5 | 440.91| 323.70| 269.92|
| CRLB                   | 215.4   | 191.9 | 171.1 | 152.5 | 135.9 | 121.1 | 107.9 |

From Table 1 and Table 2, we can conclude that the mean square error (MSE) of Ikay and ApIkay improve 3dB and 4 dB or so than that of Kay separately. From Table 2, we can see that the MSE of Kay is very large when the signal frequency is close to 1/2 sampling frequency, while Ikay and ApIkay are far better than Kay. This is because Kay’s estimator is sensitive to frequency. Now we compare the performance relationship between MSE and frequency by using three methods. Simulation conditions: SNR = 2 dB, frequency step is 5 MHz, the range of frequency is from 0 MHz to 100 MHz, there are 21 frequency points. Compute MSE of this frequency points, the result is shown in the figure 2. From Figure (2), we can see that
the performance of Kay is bad when frequency is close to 50 MHz, while Ikay and ApIky are better in the whole range of frequency. The mean MSE of Kay and Ikay and ApIky during the whole frequency are 20.0001MHz and 1358.2 KHz and 1076.5 KHz, ApIky’s is most close to CRLB which is 171.07 KHz.

Figure 2. Performance relationship between MSE and frequency by using three methods (the following figure is zoom out figure)

When SNR>6dB, the estimate accuracy of ApIky closes to CRLB. Take the above signal 20Mhz sinusoid wave as example, compute the MSE of the signal by using these three algorithms at different SNR. The result of simulation is as the Figure (3).

From Figure(3), We can draw a conclusion that ApIky reduces the SNR threshold of Kay and Ikay, that means the MSE of ApIky obtains CRLB at SNR=6dB, while Kay obtains CRLB at SNR = 10dB, Ikay obtains CRLB at SNR=8dB.

Take another signal as example to test the frequency accurate by Kay, Ikay, ApIky and ApFFT phase difference [10](named ApPD). The signal is complex exponent signal, fs is 32hz, we consider three cases about signal frequency, which is 6.7316hz, 12.2631hz, 15.7386hz separately. The SNR is 2dB. Signal length is N=256. The simulation result is as Table3.

| TABLE III. FREQUENCY ESTIMATION COMPARISON |
5. Conclusion

This paper brings forward ApFFT, which has less leakage and high precision and phase invariant compared to the traditional spectrum analysis. An improved algorithm of sinusoid wave under the noise background is proposed, which has better frequency estimation characteristic, improves the SNR threshold of Kay, and the performance is stable in the whole frequency range, which is a problem in Kay. So the research in this paper can be used in parameter estimation in the area of communication and Radar.

6. Acknowledgment

This work is supported by Science Foundation of Hebei Province (F200800015) and also supported by Youth Foundation of Education Department of Hebei Province (2010258)

7. References


