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Exponential Decay for Nonlinear Damped Equation of Suspended String

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Abstract. This paper is concerned with the energy decay of the global solution for IBVP to a nonlinear damped equation of suspended string with uniform density to which a nonlinear outer force works. For this purpose, we employed the energy method as [Wo-Ya1] and derive the decay estimate by the nonlinear damping term along the refined method of [Ma-Ik].

Keywords: nonlinear, decay estimate, suspended string, damped equation.

1. Introduction

In this work, we will study the energy decay of the global solution of a heavy and flexible string with a uniform density suspended from ceiling under gravity. Suppose that a nonlinear outer absorbing force and a nonlinear damping work to the string in horizontal direction in a vertical plane. For the derivation of the suspended string equation, see [K-G-S], [Ko] and [Ya1].

Let Ω be a cylindrical domain $(0,a)\times(0,T)$ and a be the finite length of the string. Then the above nonlinear problem is formulated as the following IBVP

$$\text{(P)}\qquad \begin{cases} \partial_t^2 u(x,t) + Lu(x,t) + \alpha\,|u_t|^{p-1} u_t = \beta\,|u|^{q-1} u, & (x,t)\in\Omega, \\ u(a,t)=0, \quad t\in(0,T), \\ u(x,0)=\phi(x), \qquad \partial_t u(x,0)=\psi(x), \qquad x\in(0,a), \end{cases}$$

where L is a second order differential operator of the form

$$L = L_0(x,\partial_x) = -(x\partial_x^2 + \partial_x),$$

and $\alpha>0, \beta\le 0$ are constants p and $q>1$.

We note that L is the special case of the differential operator L_μ (see 2) and degenerates at the origin. [Ya1] have shown the existence of almost periodic C^2-solutions to IBVP for the linear equation of suspended string equation with the quasi-periodic forcing term, and several periodic problems of the nonlinear suspended string equations have been studied in [Ya2]-[Ya4] and [Ya-Na-Ma]. [Wo-Ya2] studied the existence and uniqueness of a time-global classical solution with a monotonous cubic nonlinear term u^3, when the initial data are large. The purpose of this paper is to show decay estimate of global solution to the problem (P). Note that $\beta|u|^{q-1}u$ is the absorbing term when $\beta<0$ and linear case when $\beta=0$. In [Wo-Ya1], we proved the existence of a time-global weak solution to nonlinear equation of suspended string without the damping term by energy method based on the potential well. This research, is to show the global solution of the problem (P), Because we have the similar energy identity to the problem in [Wo-Ya1],

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therefore we can construct the potential well for the global solution by the same way as [Wo-Ya1]. To derive the decay estimate of the global solution, we apply an approach of [Ma-Ik], which discussed the global existence and energy decay to the wave equations of Kirchhoff Type with Nonlinear Damping Terms. Contrary to [Ma-Ik], we need to apply the properties of the operator L and some Poincare-Sobolev type inequalities on some appropriate function spaces suitable for our problem (see [Ya1]).

2. Function Spaces, Operator L, the Basic Inequalities

2.1. Definitions of Function Spaces

In this section, we briefly review notation and results, which will be employed later. Let R_+^1 and Z_+ be the set of nonnegative numbers and the set of nonnegative integers, respectively.

Let $p \le 1$ and $s \in Z_+$. For any open set O in R^n. $L^p(O)$ and $H^s(O)$ are the usual Lebesgue and Sobolev spaces, respectively. In the following we assume that all functions are real-valued. Let $\mu > -1$. We denote $L^p(0,a;x^\mu)$ by a Banach space whose elements $f(x)$ are measurable in $(0,a)$ and satisfy $x^{\mu/p} f(x) \in L^p(0,a)$. Its norm is

$$|f|_{L^p(0,a;x^\mu)} = \left(\int_0^a x^\mu |f(x)|^p \, dx \right)^{1/p}.$$

In particular, $L^2(0,a;x^\mu)$ is a Hilbert space with inner product

$$(f,g)_{L^p(0,a;x^\mu)} = \int_0^a x^\mu f(x) g(x) dx$$

Denote $H^s(0,a;x^\mu)$ by a Hilbert space, whose elements $f(x)$ and their weighted derivatives $x^{j/2} f^{(j)}(x),\ j = 1,...,s,$ belong to $L^2(0,a;x^\mu)$, where $f^{(j)}(x)$ means the j-th derivative of $f(x)$. Its norm

$$|f|_{H^s(0,a;x^\mu)} = \left(\sum_{j=0}^{s} \int_0^a x^{\mu+j} |f^{(j)}(x)|^2 \, dx \right)^{1/2}.$$

Denote $W^{s,p}(0,a;x^\mu)$ by a Banach space, whose elements $f(x)$ and their weighted derivatives $x^{j/p} f^{(j)}(x),\ j = 1,...,s,$ belong to $L^p(0,a;x^\mu)$. Its norm is

$$|f|_{W^{s,p}(0,a;x^\mu)} = \left(\sum_{j=0}^{s} \int_0^a x^{\mu+j} |f^{(j)}(x)|^p \, dx \right)^{1/p}.$$

Let $T > 0$ and $\Omega = (0,a) \times (0,T)$. We denote $L^p(\Omega;x^\mu)$ by a Banach space, whose elements $f(x,t)$ are measurable in Ω and satisfy $x^{\mu/p} f(x,t) \in L^p(\Omega)$. Its norm is

$$|f|_{L^p(\Omega;x^\mu)} = \left(\int_\Omega x^\mu |f(x,t)|^p \, dxdt \right)^{1/p}.$$

We denote $H^s(\Omega;x^\mu)$ by a Hilbert space, whose elements $f(x,t)$ and their weighted derivatives $x^{j/2} \partial_x^j \partial_t^k f(x,t),\ 0 \le j+k \le s,$ belong to $L^2(\Omega;x^\mu)$. Its norm is

$$|f|_{H^s(\Omega;x^\mu)} = \left(\sum_{j+k \le s} \int_\Omega x^{\mu+j} |\partial_x^j \partial_t^k f(x,t)|^2 \, dxdt \right)^{1/2}.$$

$H_0^1(0,a;x^\mu)$ is a subspace of $H^1(0,a;x^\mu)$ whose elements $f(x)$ satisfy $f(a) = 0$. Similarly, $H_0^1(\Omega;x^\mu)$ is a subspace of $H^1(\Omega;x^\mu)$ whose elements $f(x,t)$ satisfy $f(a,t) = 0$ for almost all $t \in (0,T)$. $K^s(0,a;x^\mu)$ is a subspace of $H^s(0,a;x^\mu)$ whose elements $f = f(x)$ satisfy $L_\mu^j \in H_0^1(0,a;x^\mu)$ for $j = 0,...,[(s-1)/2]$.

Note that $$K^0(0,a;x^\mu) = L^2(0,a;x^\mu), \quad K^1(0,a;x^\mu) = H_0^1(0,a;x^\mu),$$
$$K^2(0,a;x^\mu) = H^2(0,a;x^\mu) \cap H_0^1(0,a;x^\mu).$$

2.2. The Basic Properties of L_μ

We recall the more general suspended string operator L_μ introduced by [K-G-S], [Ko] for $\mu > -1$:

$$L_\mu = -(\frac{x}{\mu+1}\partial_x^2 + \partial_x).$$

Clearly, L_μ coincides with our differential operator L for $\mu = 0$.

Proposition 2.1 ([Ya1]). Let L_μ be as in (2.2) for $\mu > -1$. Then we have the following assertions: For $f \in K^2(0,a;x^\mu)$ and $g \in K^1(0,a;x^\mu)$, we have

$$(L_\mu f, g)_{L^2(0,a;x^\mu)} = \int_0^a \frac{x^{\mu+1}}{\mu+1}\partial_x f(x)\partial_x g(x)dx.$$

This lemma is the Poincare type inequality in $H_0^1(0,a;x^\mu)$.

Lemma 2.2 [Ya1]. Let $\mu > -1$. Then, for $u \in H_0^1(0,a;x^\mu)$, we have

$$|u|_{L^2(0,a;x^\mu)} \le a|\partial_x u|_{L^2(0,a;x^{\mu+1})}.$$

Lemma 2.3 [Ya2]. Let $p \ge 1$ and $\mu > -1$. Then, for $u \in W^{1,p}(0,a;x^\mu)$ we have

(2.1) $$\int_0^a x^\mu |u(x)|^p\,dx \le c_1(\int_0^a x^{\mu+1}|u(x)|^p\,dx + \int_0^a x^{\mu+p}|u'(x)|^p\,dx),$$

where the constants $c_1 > 0$ depends on μ, a, p.

Lemma 2.4 [Na]. Let $\Phi(t)$ be a non-increasing and nonnegative function on $[0,T]$, $T<1$ such that

$$\Phi(t)^{1+r} \le k_0(\Phi(t) - \Phi(t+1)) \quad \text{on } [0,T],$$

where k_0 is a positive constant, and r is a nonnegative constant. Then, we have:

(i) if we assume $r > 0$, then

$$\Phi(t) \le (\Phi(0)^{-r} + k_0 r[t-1]^+)^{-1/r} \quad \text{on } [0,T],$$

where $[t-1]^+ = \max\{t-1, 0\}$.

(ii) if $r = 0$, then $$\Phi(t) \le \Phi(0)e^{-k_1}\,[t-1]^+ \quad \text{on } [0,T],$$

where $k_1 = \log(k_0/(k_0-1))$.

3. Energy estimates and Potential well of (P)

The total energy of (P) consists of the potential energy and kinetic energy defined as follows

$$E(t;u) = \widetilde{E}(u(\cdot,t), \partial_t u(\cdot,t)) = K(u(\cdot,t)) + J(u(\cdot,t)).$$

Let $\widetilde{E}(u,u_t)$ be denoted by $$\widetilde{E}(u,u_t) = \frac{1}{2}\int_0^a u_t(x)^2\,dx + J(u).$$

The potential energy is defined as $$J(w) = \frac{1}{2}|w|^2 - \frac{\beta}{q+1}\int_0^a w(x)^{q+1}\,dx$$

for $w \in H_0^1(0,a;x^0)$, where $|w|^2 = \int_0^a x(\partial_x w(x))^2\,dx$ which is equivalent to $|\cdot|_{H_0^1}$

and the kinetic energy is defined as $$K(w) = \frac{1}{2}\int_0^a (w_t(x,t))^2\,dx \qquad \text{for } w \in H_0^1(\Omega; x^0).$$

The nonlinear term in (P) satisfies the following condition (A) [Wo-Ya1].

(A) $f(x,u)$ is of C^1-class in $(x,u) \in [0,a]\times R_u^1$ and monotone decreasing in $u \in R^1$, and satisfies

$$-C_0|u|^{r+1} \le uf(x,u) \le -C|u|^{r+1}$$

for any $x \in [0,a]$ and $u \in R^1$. Here $C_0, C > 0$ are constants and $r > 1$.

From (A), there exists a constant λ_1 such that $J(\lambda u)$ is monotone increasing in $\lambda \in (0,\lambda_1)$ for any fixed $u \ne 0$. By the same way as [Wo-Ya1], the potential well W for (P) around the origin is defined by

$$W=\{u\in H_0^1(0,a;x^0);\ 0\le J(\lambda u)<d,\ 0\le\lambda\le 1\}$$

Let $\lambda=\lambda_0(u)>0$ be the first value of λ at which $J(\lambda u)$ starts to decrease strictly. The depth d of the potential well W is defined by $d=\inf_{u\in H_0^1(0,a;x^0)\setminus\{0\}} J(\lambda_0(u)u).$

We see [Wo-Ya1] that $0<d<+\infty$ and W are open and bounded in $H_0^1(0,a;x^0)$.

We assume the following conditions on the initial data.

(B) Let $\phi\in W$ and $\psi\in L_2(0,a;x^0)$. ϕ and ψ satisfy the following condition

$$\int_0^a\left\{\frac{1}{2}\left(\psi(x)^2+x\phi'(x)^2\right)-\frac{\beta}{q+1}w(x)^{q+1}\right\}dx<d\,.$$

Main Theorem. Assume (A) and (B). Then problem (P) has a weak solution $u\in H_0^1(\Omega;x^0)$ satisfying $u(\cdot,t)\in W$ for all $t\in(0,T)$. Furthermore, we have the decay estimates:

if $p=1$, then $E(u(t),u_t(t))\le\widetilde{C}_1e^{kt}$ on $[0,+\infty)$,

and if $p>1$, then $E(u(t),u_t(t))\le\widetilde{C}_2(1+t)^{-2/(p-1)}$ on $[0,+\infty)$,

where $k,\widetilde{C}_1$ and $\widetilde{C}_2$ are any positive constants depending on initial data.

Proof Main Theorem

Multiplying eq. (P) by u_t and integrating over $[t,t+1]\times(0,a)$, we have

$$\alpha\int_0^a\int_t^{t+1}u_tu_t{}^p\,dxdt=\int_0^a\int_t^{t+1}u_tLu\,dxdt-\int_0^a\int_t^{t+1}u_tu_{tt}\,dxdt+\beta\int_0^a\int_t^{t+1}u_t\,|u|^q\,dxdt. \tag{3.1}$$

Let us consider the second term of eq.(3.3)

$$\int_t^{t+1}u_tLu\,dt=\int_t^{t+1}x\partial_xu\partial_x\left(\partial_tu\right)dt=\frac{1}{2}\int_t^{t+1}\partial_tx\left(\partial_xu\right)^2dt=\frac{1}{2}x\left(\partial_xu\right)^2\Big|_t^{t+1}. \tag{3.2}$$

We use Proposition 2.1 to obtain

$$\int_0^a\int_t^{t+1}u_tLu\,dxdt=\frac{1}{2}\int_0^a x\left(\partial_xu\right)^2\Big|_t^{t+1}dx=\frac{1}{2}|u(t+1)|^2-\frac{1}{2}|u(t)|^2\quad\text{where }|u|^2=\int_0^a x\left(\partial_xu\right)^2dx. \tag{3.3}$$

From (3.1) - (3.3), we have

$$\begin{aligned}\alpha\int_0^a\int_t^{t+1}u_t^{p+1}dxdt&=\int_0^a(u_t(t))^2\Big|_t^{t+1}dx+\frac{1}{2}|u(t)|^2-\frac{1}{2}|u(t+1)|^2-\frac{\beta}{q+1}\int_0^a u(x)^{q+1}\Big|_t^{t+1}dx\\&=E(t)-E(t+1)=\alpha D(t)^{p+1},\end{aligned} \tag{3.4}$$

when we set $E(t)=E(u,u_t)$. Multiplying eq.(P) by u and integrating over $[t_1,t_2]\times(0,a)$, we have

$$\int_{t_1}^{t_2}J(s)ds=\int_{t_1}^{t_2}\int_0^a u_t(s)^2dxds+(u(t_1),u_t(t_1)-u(t_2),u_t(t_2))-\alpha\int_{t_1}^{t_2}\left(|u_t(s)|^{p-1}u_t(s),u(s)\right)ds. \tag{3.5}$$

Note that $t_1\in[t,t+1/4]$ and $t_2\in[t+3/4,t+1]$.

By applying Holder's inequality to the second term of eq.(3.5), we have

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^a u_t(s)^2dxds&\le\int_{t_1}^{t_2}\left(\int_0^a|1|^{(p+1)/(p-1)}\,x\right)^{(p-1)/(p+1)}\left(\int_0^a u_t(s)^{\{2(p+1)\}/2}dx\right)^{2/(p+1)}ds\\&=mes(0,a)^{(p-1)/(p+1)}\int_{t_1}^{t_2}\left(\int_0^a|u_t(s)|^{(p+1)}dx\right)^{2/(p+1)}ds=mes(0,a)^{(p-1)/(p+1)}D(t)^2.\end{aligned} \tag{3.6}$$

The third term of eq.(3.5) is estimated by the mean value theorem

$$\|u_t(t_i)\|_2\le 2mes(0,a)^{p-1/\{2(p+1)\}}D(t)^2. \tag{3.7}$$

From Lemma 2.3, the last term of eq. (3.5) is estimated by

$$\begin{aligned}\alpha\int_{t_1}^{t_2}\int_0^a|u_t(s)|^p\,|u(s)|\,dxds&\le\alpha\int_{t_1}^{t_2}\left(\int_0^a|u_t(s)|^{p+1}dx\right)^{p/(p+1)}\left(\int_0^a|u(s)|^{p+1}dx\right)^{1/(p+1)}ds\\&=\alpha\int_{t_1}^{t_2}|u_t(s)|_{L^{p+1}}^p\,|u(s)|_{L^{p+1}}\,ds\end{aligned}$$

$$\leq \alpha C_1 \,|\, u_t \,|_{W^{1,p+1}} \int_{t_1}^{t_2} |u_t(s)|_{L^{p+1}}^p \, ds,$$

and let us consider the potential energy and Lemma 2.2,

$$E(t) \geq J(u(t)) = \frac{1}{2}\int_0^a x(\partial_x u(x))^2 \, dx - \frac{\beta}{q+1}\int_0^a u(x)^{q+1} dx \geq |\, u \,|_{W^{1,p+1}}$$

then we have

$$\alpha \int_{t_1}^{t_2} \int_0^a |\, u_t(s)|^p \,|\, u(s) \,|\, dxds \leq c_0 E(t)^{\frac{1}{2}} D(t)^p. \tag{3.8}$$

From (3.5) - (3.8), we obtain

$$\int_{t_1}^{t_2} J(s) ds \leq mes(0,a)^{\frac{(p-1)}{(p+1)}} D(t)^2 + 4 mes(0,a)^{\frac{(p-1)}{2(p+1)}} D(t) \sup_{t_1 \leq s \leq t_2} \|u(s)\|_2 + c_0 E(t)^{\frac{1}{2}} D(t)^p. \tag{3.9}$$

Hence, it follows from (3.6) and (3.9) that

$$\int_{t_1}^{t_2} E(s) ds = \frac{1}{2}\int_{t_1}^{t_2} |u_t(s)|_{L^2}^2 \, ds + \int_{t_1}^{t_2} J(u(s)) ds \leq c_1 (D(t)^2 + D(t) E(t)^{\frac{1}{2}} + D(t)^p E(t)^{\frac{1}{2}}). \tag{3.10}$$

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